

# ATLASES FOR INEFFECTIVE ORBIFOLDS

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**ABSTRACT.** We give a definition of atlases for ineffective orbifolds, and prove that this definition leads to the same notion of orbifold as that defined via topological groupoids.

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## 1. INTRODUCTION

This paper considers general, not necessarily effective, orbifolds; we will refer to these as ineffective. Ineffective orbifolds have been studied from several perspectives, and multiple definitions have been proposed. If we define orbifolds using topological groupoids, it is straightforward to generalize to ineffective orbifolds, and this is the approach that has often been used in the literature (for example, [ALR, CR1, CR2, FO]). On the other hand, it is nice to also have a traditional atlas definition, and here the generalization to the ineffective case becomes less obvious. Although it is easy to define an ineffective orbifold chart, figuring out what the chart embeddings are and how to build an atlas out of them is less obvious. As it stands, the ineffective atlas definitions currently used in the literature do not match the objects described using topological groupoids; this has been observed in [HM].

In this paper, we give a new definition of orbifold atlas for ineffective orbifolds, generalizing the definition of orbifold atlas in the effective case, and show that it does match the topological groupoid approach. Our definition will not change the existing notion of chart for ineffective orbifolds, but will change the structure of the embeddings between charts that create the atlas.

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The definition we develop is considerably longer and less straightforward than the one for ineffective orbifold groupoids. We believe that it is still interesting to have and that it sheds new light on the structure of ineffective orbifolds. In addition, although writing out the definition in full generality is quite long, applying it in particular examples becomes more tractable, as we will show in Example 4.15.

We see two avenues for immediate applications. The first is in considering notions of suborbifolds and embeddings. In the effective case, this has been studied from both the groupoid and atlas perspective. The original definitions were given via local charts, and it turns into a surprisingly delicate question to get the definitions correct even on this level. The original definition excludes natural examples that one would want to be included, and therefore has been refined and expanded; see for example [BB], who use the atlas and chart approach to develop examples and discuss the subtleties and various notions used here. These ideas have also been studied from a groupoid approach. Once again, the natural notion of suborbifold is very restrictive and excludes obvious examples, such as the diagonal of a product orbifold. Thus, again the idea has been expanded. For example, [CHS] develops a groupoid approach to orbifold embeddings that encompass some of the desired examples, but only manages to fully develop the theory for translation groupoids. Given this, we believe that having a fully developed notion of atlases for ineffective orbifolds is a necessary first step at considering these questions in the ineffective setting.

A second avenue of investigation comes from the idea of developing an orbifold category. This idea is inspired by the work of Grandis [G] (and further developed by Cockett et al. [CCG]), to generalize the idea of manifolds. Based on local smooth structures pieced together via join restriction categories, it is possible to define manifold objects in a variety of categories. We are interested in extending this to the idea of generalized orbifold categories, based on locally defined notions of chart and atlas.

**1.1. Organization.** The paper is structured as follows. We start by giving background on the various currently existing definitions of orbifolds, and explaining the issues surrounding the existing definitions in Section 2. In Section 3, we give some background on the language of group bimodules, which we use in defining our atlases. We give the new orbifold atlas definition in Section 4. The remainder of the paper is devoted to proving that this definition does match the orbifolds defined via topological groupoids: in Section 5 we show that we can construct an orbifold groupoid (that is, one which is étale proper and Lie) from one of our atlases, and in Section 6 we show that we can construct one of our orbifold atlases from a groupoid which defines an orbifold. In Section 7 we prove that equivalent orbifold atlases correspond to Morita equivalent groupoids, completing the correspondence between our atlas definition and the groupoid approach.

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## 2. MOTIVATION: ORBIFOLDS VIA ATLASES AND VIA GROUPOIDS

**2.1. Effective Orbifolds.** Orbifolds are spaces which are locally modeled by quotients of finite groups acting smoothly on open subsets of  $\mathbb{R}^n$ . Conceived as generalizations of manifolds, they were originally described in the same way that manifolds are, using the language of charts and atlases.

In Satake's original definition [Sa1], an orbifold chart (or uniformizing system) for an open subset  $U$  of a topological space  $X$  consists of  $\tilde{U}$ , a connected open subset of  $\mathbb{R}^n$ , with a finite group  $G$  acting *effectively* on  $\tilde{U}$ , such that the quotient space  $\tilde{U}/G$  is homeomorphic to  $U$  via the projection  $\pi$ . An embedding of charts  $(\tilde{U}_1, G_1, \pi_1) \hookrightarrow (\tilde{U}_2, G_2, \pi_2)$  is defined by a smooth embedding  $\lambda : \tilde{U}_1 \rightarrow \tilde{U}_2$  such that  $\pi_1 = \pi_2 \circ \lambda$ , so that it induces a continuous embedding on the quotient spaces. An atlas for a space then consists of a collection of charts such that the quotients cover the underlying space, with all chart embeddings between them, and suitable compatibility conditions (see Section 4.1 for full definition). We will refer to such a structure as a Satake atlas.

These atlases have several nice properties that have proved very useful when working with effective orbifolds. First, given an open subset  $\tilde{U}$  of  $\mathbb{R}^n$  and an effective action of a finite group  $G_U$  on  $\tilde{U}$ , we can think of this as an orbifold chart for  $U = \tilde{U}/G_U$ . Then the collection of chart embeddings from  $\tilde{U}$  into itself has the structure of the group  $G_U$ : each embedding  $\lambda$  is of the form  $x \mapsto g \cdot x$  for a unique  $g \in G_U$ , and each group element gives a chart embedding, such that the composition of chart embeddings corresponds to the multiplication in the group  $G_U$ . Furthermore, since for each chart embedding  $\lambda$  and each group element  $g \in G_U$  the map  $\lambda g$  is a chart embedding as well, each embedding gives rise to a unique group homomorphism  $\phi : G_U \rightarrow G_U$  given by conjugation by the element  $g$ , such that the embedding  $\lambda$  is equivariant with respect to  $\phi$  in the sense that  $\lambda(g \cdot x) = \phi(g) \cdot \lambda(x)$ . These results about the interaction of the embeddings with the structure groups were first given in [Sa2] for actions with fixed sets of codimension at least 2 and proved in full generality in [MP, Section 1 and the appendix].

More generally, these references show that if we have any chart embedding  $\lambda : \tilde{U}_1 \rightarrow \tilde{U}_2$ , we can realize this as an equivariant map: there is a unique induced homomorphism  $\phi : G_1 \rightarrow G_2$  such that  $\lambda(g \cdot x) = \phi(g) \cdot \lambda(x)$  for all  $x \in \tilde{U}_1$ .

Given any open connected subset  $V$  of a chart  $U$  with structure group  $G_U$ , it is always possible to endow  $V$  with an induced orbifold chart structure. To do this, we take a connected component  $\tilde{V}$  of the inverse image of  $V$  in the covering space  $\tilde{U}$ , and consider the action of the subgroup  $G_V$  of  $G_U$  that leaves this subset  $\tilde{V}$  invariant (but not necessarily pointwise fixed). Using this fact, a slight variation on the definition of orbifold atlas was introduced by [FO] where an atlas is given as a collection of orbifold charts which covers the space  $X$  and is *locally compatible* in the sense that all charts in the collection containing a given point  $x$  induce the same germ of a chart at that point.

Both these definitions of atlases are rather cumbersome to work with, particularly when it comes to defining maps between orbifolds with desirable properties; see [?] for instance. Therefore, an alternate description of the orbifold category has been developed using topological and differentiable groupoids. A topological groupoid has a space of objects and a space of arrows identifying certain objects; the quotient of the object space by these identifications gives the underlying quotient space of the

orbifold, and the identifications are used to encode the local singularity structure. To give a smooth structure we require our orbifold groupoids to be Lie groupoids (with smooth structures and maps). To keep the singularities of the type desired for orbifolds, that is, those given by quotients of smooth finite group actions, we place restrictions on the groupoid and ask that the diagonal be a proper map and that the groupoid be étale (i.e., such that the source and target maps from the space of arrows to the space of objects are local diffeomorphisms). When looking at effective orbifolds, we also require the groupoids to be effective. Two such groupoids represent equivalent orbifolds precisely when they are Morita equivalent [MP].

The groupoid approach and the original atlas idea lead to the same notion of *effective* orbifold. If we start with an effective orbifold atlas, we can create a groupoid by defining the object space to be a disjoint union of the chart spaces  $\tilde{U}$ , and the arrow space to consist of equivalence classes of spans of chart embeddings  $\tilde{U}_1 \leftarrow \tilde{V} \rightarrow \tilde{U}_2$  [Pro, PSi]. In particular, if we start with an orbifold atlas that consists of a single chart given by the quotient of an effective action of a finite group  $G_U$  on an open subset  $\tilde{U}$  of  $\mathbb{R}^n$ , then as discussed above, the chart embeddings from  $\tilde{U}$  to itself form a group which is isomorphic to the structure group  $G_U$ . Hence the groupoid obtained is the translation groupoid  $G_U \ltimes \tilde{U}$ , encoding the quotient orbifold as expected. It was shown in [MP] that this leads to an equivalence between the category of effective étale Lie groupoids, localized with respect to Morita equivalences, and the category of orbifolds created from effective atlases (endowed with a suitable notion of maps between orbifolds).

**2.2. Ineffective Orbifolds.** More recently, people have become interested in ineffective orbifolds, and examples of these have become important in various applications arising from mathematical physics. From a groupoid perspective it is relatively easy to define an ineffective orbifold. Just as the groupoids representing effective orbifolds are effective smooth étale groupoids with a proper diagonal, so the appropriate notion of *ineffective* orbifold would be to take a quotient of an *ineffective* smooth étale groupoid with proper diagonal (and again call two of them equivalent if there is a Morita equivalence between the groupoids). This definition of ineffective orbifold has been used frequently (see [ALR, CR1, CR2, FO] for instance) and leads immediately to a good notion of a map between ineffective orbifolds.

We can define a chart by simply dropping the effective condition from the definition given above. When we do so, however, an equivariant embedding no longer dictates a unique group homomorphism, so we need to consider what chart embeddings to include in our atlas. One generalization that has been used is to include the group homomorphism as part of the embedding information, and to define an embedding of orbifold charts as a group homomorphism  $\phi : G_1 \rightarrow G_2$  together with a map  $\lambda : \tilde{U}_1 \rightarrow \tilde{U}_2$  which is equivariant in the sense that  $\lambda(g \cdot x) = \phi(g) \cdot \lambda(x)$ , and furthermore require that  $\phi$  induces an isomorphism on the kernels of the actions [CR1].

Although we want each chart embedding to be equivariant in this sense, simply taking this as a definition of embeddings does not lead us immediately to a satisfactory definition of an orbifold atlas as a whole. In particular, we do not necessarily obtain the result (discussed above in the effective case) that the group of embeddings from a chart to itself is isomorphic to the structure group of the chart. For

example, consider an open subset  $\tilde{U}$  of  $\mathbb{R}^n$  with a trivial action of a group  $G$ , so that  $\tilde{U} = U$  and the orbifold is completely ineffective. In this case, the embedding of  $\tilde{U}$  to itself must be the identity on the space, and any automorphism  $\phi$  of the group  $G$  can be used. So if  $G$  is  $\mathbb{Z}/2$ , there are no non-trivial automorphisms of  $G$  and so there is only a single orbifold embedding from  $\tilde{U}$  to itself. On the other hand, if  $G$  is  $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ , the group of automorphisms is isomorphic to  $S_3$  rather than  $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ , and so we obtain more chart embeddings than the size of the structure group. In both of these cases, if we construct a groupoid based on the atlas of this single chart and its self-embeddings, we will *not* get the translation groupoid  $G \ltimes \tilde{U}$ . Thus, the correspondence between the atlas viewpoint and the groupoid viewpoint has broken down. This is the discrepancy observed also by Henriques and Metzler in [HM].

We can also consider whether the alternate orbifold atlas point of view, based on local compatibility, may have a better correspondence with the groupoids; this approach was used in [CR1] for instance. Unfortunately, there are smooth étale groupoids with proper diagonal which are not Morita equivalent, but whose quotient spaces are homeomorphic and whose associated atlas charts give the same germ structure at corresponding points. So these orbifolds would be the same when considered as spaces with germs of atlas charts as in [CR1], but they are not the same as ‘groupoid orbifolds’. The following example illustrates this.

**Example 2.1.** We define groupoids  $\mathcal{G}$  and  $\mathcal{H}$  representing completely ineffective orbifolds with  $\mathbb{Z}/3$  isotropy at each point on a circle  $S^1$ . Both groupoids have object space given by the circle  $S^1$ . For the first groupoid,  $\mathcal{G}$ , let the space of arrows be  $\mathbb{Z}/3 \times S^1$  (where  $\mathbb{Z}/3$  has the discrete topology) where the source and target maps are the projection,  $s(a, x) = x = t(a, x)$ . We define the groupoid composition  $m((a, x), (b, x)) = (ab, x)$ , where  $ab$  is taken as the product in  $\mathbb{Z}/3$ . For  $\mathcal{H}$  we define the space of arrows to be the disjoint union of two copies of  $S^1$ . One circle  $X$  will represent the identity arrows, with source and target maps defined by the identity  $S^1 \rightarrow S^1$ . The second circle  $Y$  will represent the non-identity arrows (which still act trivially, since this is a completely ineffective orbifold). Both the source and target maps will be the double wrapping map  $S^1 \rightarrow S^1$  that make  $Y$  into a two-fold cover of the object space  $S^1$ . Composition  $m(a, b)$  for arrows  $a$  and  $b$  is only defined when the pair of points lies in the same fiber over the space of objects (since  $s = t$ ). So let  $a, b$  be two points in the same fiber. If  $a \in X$ ,  $m(a, b) = b$  since  $X$  is the identity component, and similarly, if  $b \in X$  then  $m(a, b) = a$ . If  $a = b \in Y$ , then  $m(a, b) = c$  where  $c$  is the other point in  $Y$  in the same fiber as  $a = b$ . If  $a, b \in Y$  and  $a \neq b$ , then  $m(a, b) = c$  where  $c$  is the unique point in the identity component  $X$  in the same fiber as  $a$  and  $b$ . The reader may verify that for both groupoids the germs of the charts will give atlas charts on  $S^1$  with structure group  $\mathbb{Z}/3$ , acting ineffectively. However,  $\mathcal{G}$  and  $\mathcal{H}$  are not Morita equivalent, and so do not define the same orbifold using the groupoid definition.

**2.3. Goal of the Paper.** The goal of this paper is to give an atlas definition that does match up with the groupoid (and hence, stack) perspective of orbifolds, and agrees in a suitable sense with the classical Satake definition in the effective case. We have shown in the previous examples that the existing definitions of ineffective chart embeddings do not lead to the same objects as the groupoids, in either the Satake or local definitions of atlases. Therefore we will reconsider what a chart

embedding should be in a definition of an orbifold atlas. We will use the language of bimodules to do this, as discussed in the next section.

### 3. BACKGROUND: GROUP BIMODULES

When we create a groupoid presentation of an orbifold starting from an atlas, the structure of the groupoid depends on the collection of all the embeddings from one chart into another (or to itself). This collection will have the structure of a group bimodule, which we develop in this section.

Given any  $G$ -set  $X$  and  $H$ -set  $Y$  and any group homomorphism  $\phi : G \rightarrow H$ , we can consider the set of  $\phi$ -equivariant maps, i.e. maps  $f : X \rightarrow Y$  that satisfy  $f(g \cdot x) = \phi(g) \cdot x$ . Each such map generates further equivariant maps  $fg$  and  $hf$  for any  $g \in G$  and  $h \in H$ , giving a set of maps  $\mathbf{f}$  with commuting left  $H$ - and right  $G$ -actions. These equivariant maps form a bimodule with the left- and right-actions tying them together. (Note that we use the word ‘bimodule’ for sets with compatible left and right group actions; we do not assume that there is a notion of addition to make them abelian groups.)

Recall that in the previous attempt at defining an ineffectively atlas, a chart embedding  $(f, \phi) : (\tilde{U}, G, \pi_U) \rightarrow (\tilde{V}, H, \pi_V)$  consisted of an injective group homomorphism  $\phi : G \rightarrow H$  with a  $\phi$ -equivariant map  $f : \tilde{U} \rightarrow \tilde{V}$  between the spaces. Our approach will be to consider an equivariant chart embedding  $(f, \phi)$  as part of a larger structure, the bimodule  $\mathbf{f}$ , and to consider this entire set of maps with its left and right actions as part of the structure. Our atlas definition will be based on these bimodule chart embeddings.

**3.1. Group Bimodules.** For finite groups  $G$  and  $H$ , a bimodule  $\phi : G \rightharpoonup H$  is given by a (non-empty) set  $\mathbf{M}$ , together with a right  $G$ -action and a left  $H$ -action that are compatible, i.e., such that for  $m \in \mathbf{M}$ ,  $g \in G$  and  $h \in H$ , we have  $(h \cdot m) \cdot g = h \cdot (m \cdot g)$ . Thus, it is a set with a left  $H$ -, right  $G$ -action, which we call a left  $H$ -, right  $G$ -module, or simply a bimodule if the actions are clear from the context.

We denote the bicategory of such bimodules by **GroupMod**. In this bicategory, the identity  $G \rightharpoonup G$  is defined by the trivial bimodule  $\mathbf{G}$ , given by the set  $\mathbf{G} = G$  together with left and right actions defined by multiplication.

Composition of **GroupMod** is defined by a tensor product: given  $\phi : G \rightharpoonup H$  and  $\psi : H \rightharpoonup K$  defined by the bimodules  $\mathbf{M}$  and  $\mathbf{N}$  respectively, their composition  $\psi \circ \phi$  is given by the tensor product bimodule  $\mathbf{Q} = \mathbf{N} \otimes_H \mathbf{M} = (\mathbf{N} \times \mathbf{M}) / \sim$ , where  $\sim$  is the equivalence relation  $(y \cdot h, x) \sim (y, h \cdot x)$  for each  $x \in \mathbf{M}, y \in \mathbf{N}$  and  $h \in H$ . The actions are defined by  $k \cdot [y, x] \cdot g = [k \cdot y, x \cdot g]$  for  $g \in G$  and  $k \in K$ . We will write  $y \otimes x$  for any equivalence class  $[y, x]$  in  $\mathbf{Q}$ .

A 2-cell or natural transformation in the bicategory is defined by a map of bimodules  $\alpha : \mathbf{M} \rightarrow \mathbf{N}$ , i.e., a set map satisfying  $\alpha(h \cdot m \cdot g) = h \cdot \alpha(m) \cdot g$ . Under these definitions, **GroupMod** is a bicategory.

**3.2. Group Homomorphisms and Bimodules.** Let  $\phi : G \rightarrow H$  be a group homomorphism. We consider the induced bimodule  $\phi : G \rightharpoonup H$ . This bimodule is defined to be the set  $H$ , together with the free and transitive left action of  $H$ , and the right action of  $G$  defined by  $h \cdot g = h\phi(g)$ . Under this correspondence, the right action of  $G$  is free, respectively transitive, if and only if the original group

homomorphism  $\phi$  is injective, respectively surjective. We can think of  $\phi$  as the set consisting of maps  $G \rightarrow H$  that are of the form  $g \mapsto h\phi(g)$ .

Every bimodule  $\psi: G \twoheadrightarrow H$  for which  $H$  acts freely and transitively is of this form, although not in a canonical way. The set  $\psi$  is an  $H$ -torsor; once we choose its ‘base point’, i.e., identity element  $e$ , the group homomorphism  $\psi: G \rightarrow H$  inducing  $\psi$  can be found in the following way:  $\psi(g)$  is the unique element  $h \in H$  such that  $e \cdot g = h \cdot e$ . Note that a different choice of base point will produce a conjugate homomorphism. So we can think of these particular bimodules as corresponding to conjugacy classes of homomorphisms  $G \rightarrow H$ .

**3.3. Atlas Bimodules.** We now return to our initial consideration of how an atlas may be constructed from chart embeddings, where the full collection of embeddings from one chart to another comes with actions of the structure groups of the source and target charts. In the effective case, any two chart embeddings are always related by a unique element of the codomain structure group, so that one embedding is obtained from the other by the action of this group element [MP, Proposition A.1]. That is, the action of the codomain group is always both free and transitive on the set of embeddings. When we move to the more general ineffective setting, we will want to ensure that our chart embedding bimodules continue to have both of these important properties. Freeness of the action of the codomain group ensures that the chart embeddings can encode the orbit structure in the codomain. Transitivity ensures that all embeddings that make up the bimodule give the same maps on the quotient space.

Lastly, we consider the action of the domain group. Since the codomain action is free and transitive, we have seen in Section 3.2 that such a bimodule corresponds to a conjugacy class of group homomorphisms. For an embedding, we want these group homomorphisms to be injective, and so we require that the action of the domain structure group is free.

Therefore we make the following definition.

**Definition 3.1.** Given two finite groups  $G$  and  $H$ , a bimodule  $M: G \twoheadrightarrow H$  is an *atlas bimodule* if it has the following properties:

- it is non-empty;
- the left  $H$ -action is free and transitive;
- the right  $G$ -action is free.

It follows from the discussion in Section 3.2 that we have the following result.

**Lemma 3.2.** *For each atlas bimodule  $M: G \twoheadrightarrow H$  and for each  $m \in M$ , there is an induced injective group homomorphism  $\Lambda_m: G \rightarrow H$ , such that for each  $g \in G$  we have  $m \cdot g = \Lambda_m(g) \cdot m$ .*

Moreover, any two such group homomorphisms defined by  $M$  are related by conjugation: if  $m' \in M$  is such that  $m' = hm$ , then  $\Lambda_{m'}(g) = h\Lambda_m(g)h^{-1}$ . So the atlas bimodule defines a conjugacy class of injective group homomorphisms  $G \rightarrow H$ .

## 4. THE NEW ATLAS DEFINITION

**4.1. Satake Atlases for Effective Orbifolds.** We begin by giving the formal definition of an atlas in the effective case to use as comparison.

In Satake's original definition [Sa2], an *effective orbifold chart* (or *effective uniformizing system*) for an open subset  $U$  of a topological space  $X$  consists of a triple  $(\tilde{U}, G, \pi)$ , where:

- $\tilde{U}$  is a connected open subset of  $\mathbb{R}^n$
- $G$  is a finite group acting *effectively* on  $\tilde{U}$
- $\pi : \tilde{U} \rightarrow U$  is a continuous and surjective map that induces a homeomorphism between  $U$  and  $\tilde{U}/G$ .

An embedding of charts  $(\tilde{U}_1, G_1, \pi_1) \hookrightarrow (\tilde{U}_2, G_2, \pi_2)$  is defined by a smooth embedding  $\lambda : \tilde{U}_1 \rightarrow \tilde{U}_2$  such that  $\pi_1 = \pi_2 \circ \lambda$ . An atlas for a space consists of a collection of charts  $\mathcal{U}$  such that the quotients cover the underlying space, and the collection of all chart embeddings between them. Furthermore, the charts are required to be locally compatible in the sense that for any two charts for subsets  $U, V \subseteq X$  and any point  $x \in U \cap V$ , there is a neighbourhood  $W \subseteq U \cap V$  containing  $x$  with a chart  $(\tilde{W}, G_W, \pi_W)$  in  $\mathcal{U}$ , and chart embeddings into  $(\tilde{U}, G_U, \pi_U)$  and  $(\tilde{V}, G_V, \pi_V)$ . We will refer to such atlases, charts and chart embeddings as Satake atlases, Satake charts and Satake embeddings respectively.

It was shown in [Sa2] and [MP] that whenever  $(\tilde{U}, G_U, \pi_U)$  and  $(\tilde{V}, G_V, \pi_V)$  are effective charts with  $U \subseteq V$  and  $\tilde{U}$  is connected and simply connected, there is at least one chart embedding  $\lambda : \tilde{U} \rightarrow \tilde{V}$ . We will generally assume that our atlas charts are connected and simply connected so that we have this property.

There are a couple of additional properties of these atlases that we will use below and we will list here to make this paper more self-contained. The first result was mentioned in [Sa2] and proved in full generality in [MP].

**Lemma 4.1.** *If  $\lambda$  and  $\lambda'$  are chart embeddings  $\tilde{U} \rightarrow \tilde{V}$  and their images overlap, i.e.,  $\lambda(\tilde{U}) \cap \lambda'(\tilde{U}) \neq \emptyset$ , then there is an element  $g \in G_U$  such that  $\lambda = \lambda' \circ g$ .*

This result can be used to prove the following.

**Lemma 4.2.** *Given charts  $U_i \subseteq U_j \subseteq U_k$  with  $\tilde{U}_i$  simply connected and connected, and embeddings  $\lambda_{ki} : \tilde{U}_i \rightarrow \tilde{U}_k$  and  $\lambda_{kj} : \tilde{U}_j \rightarrow \tilde{U}_k$  such that  $\lambda_{ki}(\tilde{U}_i) \cap \lambda_{kj}(\tilde{U}_j) \neq \emptyset$ , there is an embedding  $\lambda_{ji} : \tilde{U}_i \rightarrow \tilde{U}_j$  such that  $\lambda_{kj} \circ \lambda_{ji} = \lambda_{ki}$ .*

*Proof.* Let  $x_k \in \lambda_{ki}(\tilde{U}_i) \cap \lambda_{kj}(\tilde{U}_j) \subseteq \tilde{U}_k$ . Furthermore, let  $x_i \in \tilde{U}_i$  and  $x_j \in \tilde{U}_j$  be the unique points such that  $\lambda_{ki}(x_i) = x_k$  and  $\lambda_{kj}(x_j) = x_k$ . Now let  $\nu : \tilde{U}_i \rightarrow \tilde{U}_j$  be an embedding such that  $\nu(x_i) = x_j$ . Then  $\lambda_{kj} \circ \nu(\tilde{U}_i) \cap \lambda_{ki}(\tilde{U}_i) \neq \emptyset$ . By Lemma 4.1 there is an element  $g \in G_i$  such that  $\lambda_{kj} \circ \nu \circ g = \lambda_{ki}$ . So  $\lambda_{ji} := \nu \circ g$  has the required property.  $\square$

The third result can be found in [Pro] and [PSi] and is the key result in the construction of the effective groupoid associated to a Satake atlas. It is a stronger version of local compatibility property of charts in atlases, called *strong local compatibility*, which follows from the atlas properties:

**Lemma 4.3.** *Given charts  $\tilde{U}_1, \tilde{U}_2, \tilde{U}_3$  in an atlas  $\mathcal{U}$ , with chart embeddings  $\lambda_{31} : \tilde{U}_1 \rightarrow \tilde{U}_3$  and  $\lambda_{32} : \tilde{U}_2 \rightarrow \tilde{U}_3$  and points  $x_1 \in \tilde{U}_1$  and  $x_2 \in \tilde{U}_2$  with  $\lambda_{31}(x_1) = \lambda_{32}(x_2)$ , there is a chart  $\tilde{U}_4$  in  $\mathcal{U}$  with embeddings  $\lambda_{i4} : \tilde{U}_4 \rightarrow \tilde{U}_i$  for  $i = 1, 2$ , such that  $\lambda_{31} \circ \lambda_{i4} = \lambda_{32} \circ \lambda_{24}$ . Moreover, there is a point  $y$  in  $\tilde{U}_4$ , such that  $\lambda_{i4}(y) = x_i$  for  $i = 1, 2$ .*



**4.2. Ineffective Atlas Definition.** We give our new definition for an orbifold atlas in the language of atlas bimodules of the previous section. We will use the notion of chart for ineffective orbifolds that has become standard in the literature.

**Definition 4.4.** [ALR, CR1, CR2, LU] Let  $U$  be a non-empty connected topological space; an *orbifold chart* (also known as a *uniformizing system*) of dimension  $n$  for  $U$  is a quadruple  $(\tilde{U}, G, \rho, \pi)$  where:

- $\tilde{U}$  is a connected and simply connected open subset of  $\mathbb{R}^n$ ;
- $G$  is a finite group;
- $\rho : G \rightarrow \text{Aut}(\tilde{U})$  is a (not necessarily faithful) representation of  $G$  as a group of smooth automorphisms of  $\tilde{U}$ ; we set  $G^{\text{red}} := \rho(G) \subseteq \text{Aut}(\tilde{U})$  and  $\text{Ker}(G) := \text{Ker}(\rho) \subseteq G$ ;
- $\pi : \tilde{U} \rightarrow U$  is a continuous and surjective map that induces a homeomorphism between  $U$  and  $\tilde{U}/G^{\text{red}}$ .

Associated to every orbifold chart  $(\tilde{U}, G, \rho, \pi)$  we have the Satake chart  $(\tilde{U}, G^{\text{red}}, \pi)$  with the reduced group  $G^{\text{red}}$ ; we will refer to this also as the *reduced chart*. An orbifold atlas  $\mathcal{U}$  for a space  $X$  will contain a collection  $\mathcal{U}$  of such orbifold charts such that the underlying quotient spaces are open in  $X$  and cover all of  $X$ , and in fact, we will require that the associated reduced charts form a Satake atlas.

We write  $\mathcal{O}(X)$  for the lattice of all open subsets of  $X$  and given an atlas  $\mathcal{U}$  we will define  $\mathcal{O}(\mathcal{U})$  to be the category with objects the charts of  $\mathcal{U}$ , with morphisms given by inclusions of the underlying quotient sets. The poset  $\mathcal{O}(\mathcal{U})$  is not required to be a lattice, as it is obvious that it is not closed under unions, and contrary to the case of manifolds it may not be closed under intersections either.

The usual local compatibility condition for Satake atlases requires that for any two charts  $U$  and  $V$  and point  $x \in U \cap V$ , there is a chart  $W \subseteq U \cap V$  containing  $x$ . This can be phrased in terms of the elements of the posets  $\mathcal{O}(X)$  and  $\mathcal{O}(\mathcal{U})$  as follows:

$$\bigcup \{W \in \mathcal{O}(\mathcal{U}); W \subseteq U, W \subseteq V\} = U \cap V \quad \text{in } \mathcal{O}(X).$$

We will still require this for ineffective orbifolds. We will also consider these posets as categories with at most one arrow between any two objects: we write  $\mu_{VU} : U \rightarrow V$  whenever  $U \subseteq V$ . When we are working with an indexed family of charts  $U_i$  with  $i \in I$ , we will write  $\mu_{ji}$  for  $\mu_{U_j U_i}$ .

For a Satake atlas with connected, simply connected charts, whenever there is an arrow  $\mu_{ji} : U_i \rightarrow U_j$  between the quotient open subsets of two orbifold charts, there is a Satake embedding  $\lambda_{ji} : \tilde{U}_i \rightarrow \tilde{U}_j$  (i.e., with  $\pi_i = \pi_j \circ \lambda_{ji}$ ). We will also call these embeddings *concrete*. We will show that the set of all these embeddings form an atlas bimodule  $G_i^{\text{red}} \dashrightarrow G_j^{\text{red}}$ .

**Definition 4.5.** Fix orbifold charts  $(\tilde{U}_1, G_1, \rho_1, \pi_1)$  and  $(\tilde{U}_2, G_2, \rho_2, \pi_2)$  for open sets  $U_1$  and  $U_2$  of a given topological space  $X$ , with  $U_1 \subseteq U_2$ . Then we define the set of *concrete embeddings*

$$\text{Con}(\mu_{21}) := \left\{ \text{all smooth embeddings } \lambda : \tilde{U}_1 \rightarrow \tilde{U}_2 \text{ s.t. } \pi_1 = \pi_2 \circ \lambda \right\}.$$

This set coincides with the set of all Satake embeddings from the reduced chart  $(\tilde{U}_1, G_1^{\text{red}}, \pi_1)$  to the reduced chart  $(\tilde{U}_2, G_2^{\text{red}}, \pi_2)$ .

**Lemma 4.6.** *Let  $X$  be a topological space with a collection of orbifold charts  $\mathcal{U} = \{(\tilde{U}_i, G_i, \rho_i, \pi_i)\}_{i \in I}$  of dimension  $n$ , such that the reduced charts  $\{(\tilde{U}_i, G_i^{\text{red}}, \pi_i)\}_{i \in I}$  form a Satake atlas  $\mathcal{U}^{\text{red}}$ . For any  $\mu_{ji}$  in  $\mathcal{O}(\mathcal{U})$ , the set  $\text{Con}(\mu_{ji})$  forms an atlas bimodule  $G_i^{\text{red}} \dashrightarrow G_j^{\text{red}}$  with actions given by composition. We will denote this bimodule by*

$$\text{Con}(\mu_{ji}) : G_i^{\text{red}} \dashrightarrow G_j^{\text{red}}.$$

*Furthermore, if  $i = j$  the atlas bimodule  $\text{Con}(\mu_{ii})$  is isomorphic to the trivial bimodule  $G_i^{\text{red}}$  associated to the group  $G_i^{\text{red}}$ .*

*Proof.* As noted by Satake [Sa2] and proved in full generality in [MP, Proposition A.1], for any two atlas embeddings between effective charts  $\lambda, \lambda' : \tilde{U}_i \rightrightarrows \tilde{U}_j$ , there is a unique group element  $\rho_j(g) \in G_j^{\text{red}}$  such that  $\rho_j(g) \circ \lambda = \lambda'$ . So the action by  $G_j^{\text{red}}$  is free and transitive. The fact that the action by  $G_i^{\text{red}}$  on this set is free follows from the fact that its action on  $\tilde{U}_i$  is effective.  $\square$

For Satake orbifolds, and hence for the reduced charts of ineffective orbifolds, this family of bimodules is compatible in the following sense:

**Lemma 4.7.** *Let  $X$  be a topological space with a collection of orbifold charts  $\mathcal{U} = \{(\tilde{U}_i, G_i, \rho, \pi_i)\}_{i \in I}$  of dimension  $n$ , such that the reduced charts  $\{(\tilde{U}_i, G_i^{\text{red}}, \pi_i)\}_{i \in I}$  form a Satake atlas  $\mathcal{U}^{\text{red}}$ . Then there is a pseudofunctor*

$$\text{Con} : \mathcal{O}(\mathcal{U}) \longrightarrow \text{GroupMod},$$

*defined on objects by  $\text{Con}(U_i) := G_i^{\text{red}}$  and on morphisms by the atlas bimodules*

$$\text{Con}(\mu_{ji}) : G_i^{\text{red}} \dashrightarrow G_j^{\text{red}}$$

*described above.*

*Proof.* In order to make  $\text{Con}$  a pseudofunctor, we need to equip it with a coherent family of composition and unit 2-cells. Specifically, we need natural isomorphisms of bimodules  $\gamma_{kji} : \text{Con}(\mu_{kj}) \circ \text{Con}(\mu_{ji}) \Rightarrow \text{Con}(\mu_{ki})$  for each composable pair of arrows  $U_i \xrightarrow{\mu_{ji}} U_j \xrightarrow{\mu_{kj}} U_k$  in  $\mathcal{O}(\mathcal{U})$ , and natural isomorphisms  $\gamma_i$  from  $G_i^{\text{red}}$  to  $\text{Con}(\mu_{ii})$ , where  $G_i^{\text{red}}$  is the trivial bimodule which represents the identity arrow in  $\text{GroupMod}$  for  $G_i^{\text{red}}$ .

Recall that  $\text{Con}(\mu_{kj}) \circ \text{Con}(\mu_{ji})$  is given by the atlas bimodule  $\text{Con}(\mu_{kj}) \otimes_{G_j^{\text{red}}} \text{Con}(\mu_{ji})$ . We define  $\gamma_{kji}$  first as a set map from  $\text{Con}(\mu_{kj}) \otimes_{G_j^{\text{red}}} \text{Con}(\mu_{ji})$  to  $\text{Con}(\mu_{ki})$  by taking the equivalence class  $\lambda_{kj} \otimes \lambda_{ji}$  to the composition  $\lambda_{kj} \circ \lambda_{ji} \in \text{Con}(\mu_{ki})$  (where  $\lambda_{kj} \in \text{Con}(\mu_{kj})$  and  $\lambda_{ji} \in \text{Con}(\mu_{ji})$ ). It is easy to see that this is a well defined map; moreover, using [MP, Proposition A.1] and Lemma 4.1 we get that  $\gamma_{kji}$  is a bijection compatible with the actions of  $G_i^{\text{red}}$  and  $G_k^{\text{red}}$ , i.e., it corresponds to an invertible 2-cell in  $\text{GroupMod}$  as desired.

The  $\gamma_i : G_i^{\text{red}} \Rightarrow \text{Con}(\mu_{ii})$  are given by the isomorphisms mentioned in Lemma 4.6.

A straightforward computation shows that the data of the  $\gamma_{kji}$  for each composable pair of arrows  $U_i \xrightarrow{\mu_{ji}} U_j \xrightarrow{\mu_{kj}} U_k$  in  $\mathcal{O}(\mathcal{U})$ , the  $\gamma_i$  for each object  $U_i$  in  $\mathcal{O}(\mathcal{U})$ , together with the data of  $\text{Con}(U_i)$  and  $\text{Con}(\mu_{ji})$  form a pseudofunctor  $\text{Con}$  from  $\mathcal{O}(\mathcal{U})$  to  $\text{GroupMod}$ .  $\square$

Note that  $\mathbf{Con}$  cannot be defined as a strict functor, since the sets  $\mathbf{Con}(\mu_{kj}) \otimes_{G_j^{\text{red}}} \mathbf{Con}(\mu_{ji})$  and  $\mathbf{Con}(\mu_{ki})$  are only isomorphic, not equal. Furthermore, the isomorphism  $\gamma_i: G_i \Rightarrow \mathbf{Con}(\mu_{ii})$  for each  $U_i \in \mathcal{O}(\mathcal{U})$  provides us with a kind of base point for each  $\mathbf{Con}(\mu_{ii})$ , picking out a unit element  $\gamma_i(e_i^{\text{red}})$  for the  $G_i^{\text{red}}$ -torsor  $\mathbf{Con}(\mu_{ii})$  (where we write  $e_i^{\text{red}}$  for the unit element of  $G_i^{\text{red}}$ ).

The pseudofunctor  $\mathbf{Con}$  will form one layer of our atlas, giving the information about the structure on the quotient space level and the information about the reduced structure underlying the (possibly) non-reduced one. However, as discussed above and seen in our examples, when the action is not effective we may need more atlas embeddings in order to have our atlas correctly encode the isotropy of the codomain charts. We will therefore create another layer of bimodules to our atlas embeddings, which we will call the bimodules of *abstract embeddings*. These will be atlas bimodules for the full structure groups; that is, the action will be free and transitive for the entire codomain structure group  $G_j$ , and free for the entire domain structure group  $G_i$  (and not just the reduced groups as above). For every pair of charts  $\tilde{U}_i, \tilde{U}_j$  with  $U_i \subseteq U_j$  there will be a surjection from the bimodule of abstract embeddings to the bimodule of concrete embeddings. So each abstract embedding will correspond to a concrete one, but some bimodules of abstract embeddings may be larger, indicating an ineffective action.

**Definition 4.8.** Let  $X$  be a paracompact, second countable, Hausdorff topological space. An *orbifold atlas* of dimension  $n$  for  $X$ , denoted  $\mathfrak{U}$ , consists of the datum of:

- (1) a collection  $\mathcal{U} = \{(\tilde{U}_i, G_i, \rho_i, \pi_i)\}_{i \in I}$  of orbifold charts, of dimension  $n$ , connected and simply connected, such that the reduced charts  $\{(\tilde{U}_i, G_i^{\text{red}}, \pi_i)\}_{i \in I}$  form a Satake atlas for  $X$ ; we denote by  $(\mathbf{Con}, \gamma): \mathcal{O}(\mathcal{U}) \rightarrow \mathbf{GroupMod}$  the induced pseudofunctor as in Lemma 4.7;
- (2) a pseudofunctor

$$\mathbf{Abst}: \mathcal{O}(\mathcal{U}) \longrightarrow \mathbf{GroupMod}$$

such that for each  $i \in I$ ,  $\mathbf{Abst}(U_i) = G_i$  and for each  $\mu_{ji}$  in  $\mathcal{O}(\mathcal{U})$ ,  $\mathbf{Abst}(\mu_{ji})$  is an atlas bimodule  $G_i \dashrightarrow G_j$ , (i.e., the left action of  $G_j$  is free and transitive and the right action of  $G_i$  is free). We denote the pseudofunctor composition isomorphisms of  $\mathbf{Abst}$  by

$$\alpha_{kji}: \mathbf{Abst}(\mu_{kj}) \circ \mathbf{Abst}(\mu_{ji}) \Longrightarrow \mathbf{Abst}(\mu_{ki})$$

for each composable pair of arrows  $U_i \xrightarrow{\mu_{ji}} U_j \xrightarrow{\mu_{kj}} U_k$  in  $\mathcal{O}(\mathcal{U})$  and the identity isomorphisms by

$$\alpha_i: G_i \Longrightarrow \mathbf{Abst}(\mu_{ii}) \quad \text{for each } i \in I;$$

- (3) an oplax transformation  $\boldsymbol{\rho} = (\{\rho_i\}_{i \in I}, \{\rho_{ji}\}_{i,j \in I, U_i \subseteq U_j}): \mathbf{Abst} \Rightarrow \mathbf{Con}$ : Recall that each  $\rho_i$  is a group homomorphism from  $G_i$  to  $G_i^{\text{red}}$ , hence it induces a bimodule  $\boldsymbol{\rho}_i: G_i \dashrightarrow G_i^{\text{red}}$  (as in Section 3.2). We require that these  $\boldsymbol{\rho}_i$  be the components of an oplax transformation  $\boldsymbol{\rho}: \mathbf{Abst} \Rightarrow \mathbf{Con}$ . So, for each arrow  $U_i \xrightarrow{\mu_{ji}} U_j$  in  $\mathcal{O}(\mathcal{U})$ , there is a map of bimodules,

$$\rho_{ji}: \boldsymbol{\rho}_j \otimes_{G_j} \mathbf{Abst}(\mu_{ji}) \Longrightarrow \mathbf{Con}(\mu_{ji}) \otimes_{G_i^{\text{red}}} \boldsymbol{\rho}_i,$$

as in

$$\begin{array}{ccc}
G_i & \xrightarrow{\text{Abst}(\mu_{ji})} & G_j \\
\rho_i \downarrow & \searrow \rho_{ji} & \downarrow \rho_j \\
G_i^{\text{red}} & \xrightarrow{\text{Con}(\mu_{ji})} & G_j^{\text{red}}
\end{array}$$

We further require that:

- (a) the  $\rho_{ji}$  are surjective maps of bimodules;
- (b) (transitivity on the kernel) whenever  $\rho_{ji}(e_j^{\text{red}} \otimes \lambda) = \rho_{ji}(e_j^{\text{red}} \otimes \lambda')$  for  $\lambda, \lambda' \in \text{Abst}(\mu_{ji})$ , there is an element  $g \in G_i$  such that  $\lambda \cdot g = \lambda'$  (here  $e_j^{\text{red}}$  is the identity element of  $G_j^{\text{red}}$ ).

For simplicity, we may denote the atlas  $\mathcal{U}$  by  $(X, \mathcal{U}, \text{Abst}, \rho)$  or simply  $(\mathcal{U}, \text{Abst}, \rho)$  if  $X$  is clear from the context.

**Notation 4.9.** As in the previous definition, we will use the notation  $e_i$  for the unit element of  $G_i$  and  $e_i^{\text{red}}$  for the unit element of  $G_i^{\text{red}}$ . However, when the group is clear from the context we will simply write  $e$  for the unit element.

**Remark 4.10.** The 2-cell components of an oplax transformation between pseudo-functors are required to be compatible with the structure cells of the pseudofunctors. Since we will use this property several times in the rest of this paper, we spell it out explicitly for the oplax transformation  $\rho$ .

For any composable pair of arrows  $U_i \xrightarrow{\mu_{ji}} U_j \xrightarrow{\mu_{kj}} U_k$  in  $\mathcal{O}(\mathcal{U})$ ,

$$\begin{array}{ccc}
& G_j & \\
\text{Abst}(\mu_{ji}) \nearrow & & \searrow \text{Abst}(\mu_{kj}) \\
G_i & & G_k \\
\downarrow \rho_i & \text{Abst}(\mu_{ki}) & \downarrow \rho_k \\
G_i^{\text{red}} & \xrightarrow{\text{Con}(\mu_{ki})} & G_k^{\text{red}}
\end{array}
\quad \equiv \quad
\begin{array}{ccc}
& G_j & \\
\text{Abst}(\mu_{ji}) \nearrow & & \searrow \text{Abst}(\mu_{kj}) \\
G_i & & G_k \\
\downarrow \rho_i & \downarrow \rho_j & \downarrow \rho_k \\
G_i^{\text{red}} & \xrightarrow{\text{Con}(\mu_{ji})} G_j^{\text{red}} \xrightarrow{\text{Con}(\mu_{kj})} & G_k^{\text{red}}
\end{array}$$

For any  $U_i$  in  $\mathcal{O}(\mathcal{U})$ ,

$$\begin{array}{ccc}
& G_i & \\
& \downarrow \alpha_i & \\
G_i & \xrightarrow{\text{Abst}(\mu_{ii})} & G_i \\
\downarrow \rho_i & \searrow \rho_{ii} & \downarrow \rho_i \\
G_i^{\text{red}} & \xrightarrow{\text{Con}(\mu_{ii})} & G_i^{\text{red}}
\end{array}
\quad \equiv \quad
\begin{array}{ccc}
& G_i & \\
& \downarrow \gamma_i & \\
G_i & \xrightarrow{G_i^{\text{red}}} & G_i \\
\downarrow \rho_i & \searrow \hat{\rho}_i & \downarrow \rho_i \\
G_i^{\text{red}} & \xrightarrow{\text{Con}(\mu_{ii})} & G_i^{\text{red}}
\end{array}$$

where  $\hat{\rho}_i$  is the transformation determined by  $\hat{\rho}_i(e \otimes g) := \rho_i(g) \otimes e$ .

**Remark 4.11.** Satake did not require in his original definition of orbifold atlas that there be an atlas embedding (a Satake embedding in our terminology)  $\tilde{U} \hookrightarrow \tilde{V}$  whenever  $U \subseteq V$  in the underlying space. Instead, he required that for each point  $x \in U \cap V$  there be a chart  $W$  containing  $x$  with chart embeddings  $\tilde{W} \hookrightarrow \tilde{U}$  and  $\tilde{W} \hookrightarrow \tilde{V}$ . However, as shown in [MP], when we require that the charts be connected and simply connected, this condition implies that there is a Satake embedding  $\tilde{U} \hookrightarrow \tilde{V}$  whenever  $U \subseteq V$  in the underlying space.

So when we set up the framework for our definition of orbifold atlas we had the choice to work with more general charts and a slightly weaker compatibility condition or to work with simply connected charts. It is more work to prove in this context, but if one were given an atlas with only the weaker compatibility condition (so we would take  $\mathcal{O}(\mathcal{U})$  to be a subcategory of  $\mathcal{O}(X)$  which is not necessarily a full subcategory, but still satisfies a weaker condition) where the charts were connected and simply connected, then it would be possible to extend **Con** and **Abst** to pseudofunctors on the full subcategory and extend  $\rho$  to a pseudonatural transformation between them. This result can be found in forthcoming work by Sibih [Si]. In order to make the arguments on the relationship between orbifold atlases and groupoids run more smoothly we have chosen to use the slightly stronger compatibility condition as stated in the definition above, but all results still work when one works with the weaker condition.

We now show the connection between our new atlas definition and the previously defined version discussed in Section 2.

**Lemma 4.12.** *Each element  $\lambda \in \text{Abst}(\mu_{ji})$  defines a concrete chart embedding  $\tilde{\rho}_{ji}(\lambda) = \tilde{\lambda}: \tilde{U}_i \rightarrow \tilde{U}_j$  and a group homomorphism  $\ell_\lambda: G_i \rightarrow G_j$  such that  $\ell_\lambda$  restricts to an isomorphism on the kernels and  $\tilde{\lambda}$  is equivariant with respect to  $\ell_\lambda$  in the sense that*

$$\tilde{\lambda}(\rho_i(g) \cdot x) = \rho_j(\ell_\lambda(g)) \cdot \tilde{\lambda}(x),$$

for all  $x \in \tilde{U}_i$  and  $g \in G_i$ .

*Proof.* The map of bimodules  $\rho_{ji}: \rho_j \otimes_{G_j} \text{Abst}(\mu_{ji}) \Rightarrow \text{Con}(\mu_{ji}) \otimes_{G_i^{\text{red}}} \rho_i$  is a surjective map of sets which is equivariant with respect to the actions of  $G_i$  and  $G_j^{\text{red}}$ . The underlying set  $S$  of  $\rho_j \otimes_{G_j} \text{Abst}(\mu_{ji})$  consists of elements of the form  $g \otimes \lambda$  where  $g \in G_j^{\text{red}}$  and  $\lambda \in \text{Abst}(\mu_{ji})$ ; and for any  $k \in G_j$  we have that  $g \cdot \rho_j(k) \otimes \lambda = g \otimes k \cdot \lambda$ . Since  $\rho_j$  is surjective onto  $G_j^{\text{red}}$ , this means that each element of  $S$  can be written in the form  $e \otimes \lambda$ . However, this representation is generally not unique: whenever  $k \in \text{Ker}(\rho_j)$ , we have that  $e \otimes k \cdot \lambda = e \otimes \lambda$ .

Similarly, each element of the underlying set of  $\text{Con}(\mu_{ji}) \otimes_{G_i^{\text{red}}} \rho_i$  can be represented as  $\mu \otimes e$  for some  $\mu \in \text{Con}(\mu_{ji})$ . Here the representation is unique:  $\mu \otimes e = \mu' \otimes e$  if and only if  $\mu = \mu'$ . This allows us to define a map  $\tilde{\rho}_{ji}: \text{Abst}(\mu_{ji}) \rightarrow \text{Con}(\mu_{ji})$  by

$$(4.1) \quad \rho_{ji}(e \otimes \lambda) = \tilde{\rho}_{ji}(\lambda) \otimes e.$$

We write  $\tilde{\lambda}$  for  $\tilde{\rho}_{ji}(\lambda)$ . So  $\tilde{\lambda}$  is the unique element of  $\text{Con}(\mu_{ji})$  such that  $\rho_{ji}(e \otimes \lambda) = \tilde{\lambda} \otimes e$ .

Note that if  $k \in \text{Ker}(\rho_j)$ , we have that  $\widetilde{\lambda} = \widetilde{k \cdot \lambda}$ . Also, the map  $\widetilde{\rho}_{ji}$  is equivariant with respect to  $\rho_i$  and  $\rho_j$ : For  $g \in G_i$ ,

$$\begin{aligned} \widetilde{\lambda \cdot g} \otimes e &= \rho_{ji}(e \otimes (\lambda \cdot g)) \\ &= \rho_{ji}((e \otimes \lambda) \cdot g) \\ &= \rho_{ji}(e \otimes \lambda) \cdot g \\ &= (\widetilde{\lambda} \otimes e) \cdot g \\ &= \widetilde{\lambda} \otimes g \\ &= (\widetilde{\lambda} \cdot \rho_i(g)) \otimes e \end{aligned}$$

so

$$\widetilde{\lambda \cdot g} = \widetilde{\lambda} \cdot \rho_i(g).$$

A similar argument shows that for  $h \in G_j$ ,

$$\widetilde{h \cdot \lambda} = \rho_j(h) \cdot \widetilde{\lambda}.$$

Now let  $\lambda \in \text{Abst}(\mu_{ji})$  be arbitrary and let  $\ell_\lambda: G_i \rightarrow G_j$  be the induced group homomorphism defined by  $\lambda \cdot g = \ell_\lambda(g) \cdot \lambda$  as in Lemma 3.2. Note that  $\ell_\lambda$  sends the kernel of  $\rho_i$  to the kernel of  $\rho_j$ : If  $k \in \text{Ker}(\rho_i)$ , then  $\widetilde{\rho}_{ji}(\lambda \cdot k) = \widetilde{\rho}_{ji}(\lambda) \cdot \rho_i(k) = \widetilde{\rho}_{ji}(\lambda)$ . Hence,  $\rho_j(\ell_\lambda(k)) \cdot \widetilde{\rho}_{ji}(\lambda) = \widetilde{\rho}_{ji}(\ell_\lambda(k) \cdot \lambda) = \widetilde{\rho}_{ji}(\lambda)$ . So  $\rho_j(\ell_\lambda(k)) \in \text{Ker}(\rho_j)$ .

Furthermore,  $\ell_\lambda$  is also surjective on the kernels: Let  $h \in \text{Ker}(\rho_j)$ . Then  $\rho_{ji}(e \otimes h \cdot \lambda) = \rho_{ji}(\rho_j(h) \otimes \lambda) = \rho_{ji}(e \otimes \lambda)$ , so by condition (3b) of Definition 4.8, there is an element  $g \in G_i$  such that  $h \cdot \lambda = \lambda \cdot g$ . Hence,  $\ell_\lambda(g) = h$ . We check that  $g \in \text{Ker}(\rho_i)$ :

$$\begin{aligned} \widetilde{\rho}_{ji}(\lambda) \cdot \rho_i(g) &= \widetilde{\rho}_{ji}(\lambda \cdot g) \\ &= \widetilde{\rho}_{ji}(h \cdot \lambda) \\ &= \rho_j(h) \cdot \widetilde{\rho}_{ji}(\lambda) \\ &= \widetilde{\rho}_{ji}(\lambda). \end{aligned}$$

Since the action of  $G_i^{\text{red}}$  is free on  $\widetilde{U}_i$ , this implies that  $\rho_i(g) = e$ .

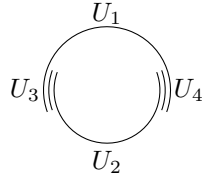
Since the homomorphism  $\ell_\lambda$  is injective by Lemma 3.2 we conclude that it restricts to an isomorphism on the kernels.  $\square$

- Remarks 4.13.** (1) The operation  $(\widetilde{\phantom{x}})$  of taking the underlying concrete embedding commutes with composition in the sense that for  $\lambda \in \text{Abst}(\mu_{ji})$  and  $\nu \in \text{Abst}(\mu_{kj})$ ,  $(\alpha_{kji}(\widetilde{\nu \otimes \lambda})) = \gamma_{kji}(\widetilde{\nu}, \widetilde{\lambda}) = \widetilde{\nu \circ \lambda}$ . We can see this as follows. The compatibility of the  $\rho_{ji}$  with the  $\alpha_{kji}$  and  $\gamma_{kji}$  from Remark 4.10 gives us that  $\rho_{ki}(e \otimes \alpha_{kji}(\nu \otimes \lambda)) = \gamma_{kji}(\widetilde{\nu} \otimes \widetilde{\lambda}) \otimes e$ . So by identity (4.1) we get  $(\alpha_{kji}(\widetilde{\nu \otimes \lambda})) = \gamma_{kji}(\widetilde{\nu} \otimes \widetilde{\lambda})$ . But  $\gamma_{kji}$  is just the operation of taking the composition, so we get  $(\alpha_{kji}(\widetilde{\nu \otimes \lambda})) = \widetilde{\nu \circ \lambda}$ , as claimed. By definition of  $\widetilde{\rho}_{ki}$ , this is equivalent to saying that  $\widetilde{\rho}_{ki}(\alpha_{kji}(\nu \otimes \lambda)) = \widetilde{\nu \circ \lambda}$ .
- (2) Since each  $\rho_{ji}$  is surjective, the map  $\widetilde{\rho}_{ji}$  sending  $\lambda \in \text{Abst}(\mu_{ji})$  to  $\widetilde{\lambda} \in \text{Con}(\mu_{ji})$  is also surjective.
- (3) Using condition (3b) in Definition 4.8 and identity (4.1), if  $\widetilde{\rho}_{ji}(\lambda) = \widetilde{\rho}_{ji}(\lambda')$ , then there is an element  $g \in G_i$  such that  $\lambda \cdot g = \lambda'$ .

**Remarks 4.14.** Any Satake atlas gives rise to a canonical atlas as in Definition 4.8: we simply take  $\text{Abst}(\mu_{ji}) = \text{Con}(\mu_{ji})$  and  $\rho_{ji}$  is defined by  $\rho_{ji}(e_j \otimes \lambda) = \lambda \otimes e_i$ . It is possible that the  $\rho_{ji}$  may be given by different isomorphisms, instead of the ones just described. However, each effective atlas is isomorphic to one coming from a Satake atlas via a canonical isomorphism.

**Example 4.15.** In order to make these ideas more concrete, we briefly describe the ineffective orbifold atlas structures for the two  $\mathbb{Z}/3$  orbigroupoids  $\mathcal{G}$  and  $\mathcal{H}$  described in Example 2.1.

Both orbifolds consist of completely ineffective actions on the circle  $S^1$ . So for both atlas structures, we may take four charts to cover  $S^1$ , one for the upper semicircle and one for the lower (here denoted  $U_1$  and  $U_2$ ), and two smaller charts embedding into the overlaps on the left and right sides (denoted  $U_3$  and  $U_4$ ).



For each chart  $U_i$ , we have  $G_{U_i} = \mathbb{Z}/3$  and  $G_{U_i}^{\text{red}} = \{e\}$ . We also have inclusions  $\lambda_{13} : U_3 \hookrightarrow U_1$ ,  $\lambda_{23} : U_3 \hookrightarrow U_2$ ,  $\lambda_{14} : U_4 \hookrightarrow U_1$  and  $\lambda_{24} : U_4 \hookrightarrow U_2$ . So for each inclusion  $\mu_{ji} : U_i \hookrightarrow U_j$ , we need to define a module  $M_{ji}$  (with compatible left free, transitive action by  $G_{U_j}$  and right free action by  $G_{U_i}$ ) and a map of bimodules  $\rho_{ji}$  as follows:

$$(4.2) \quad \begin{array}{ccc} \mathbb{Z}/3 = \{e, \omega_i, \omega_i^2\} & \xrightarrow{\text{Abst}(\mu_{ji})=M_{ji}} & \mathbb{Z}/3 = \{e, \omega_j, \omega_j^2\} \\ \rho_i \downarrow & \swarrow \rho_{ji} & \downarrow \rho_j \\ \{e\} & \xrightarrow{\text{Con}(\mu_{ji})=\{\lambda_{ji}\}} & \{e\} \end{array}$$

Using the description given in the proof of Lemma 4.12, in this case both the source and the target of  $\rho_{ji}$  are a module with one element and trivial actions, so we define  $\rho_{ji}$  as the unique isomorphism. Since the left action of  $G_{U_j}$  must be free and transitive, each  $M_{ji}$  must consist of three objects, say  $a_{ji}, b_{ji}, c_{ji}$ .

In order to create an atlas for the orbifold  $\mathcal{G}$  represented by the groupoid with three disjoint circles in the arrow space, we define the various bimodules  $M_{ji}$  by giving the following actions:

$$(4.3) \quad \begin{array}{ll} \text{left multiply by } \omega_j & \begin{array}{c} \curvearrowright \\ a_{ji} \longrightarrow b_{ji} \longrightarrow c_{ji} \end{array} \\ \text{right multiply by } \omega_i & \begin{array}{c} \curvearrowright \\ a_{ji} \longrightarrow b_{ji} \longrightarrow c_{ji} \end{array} \end{array}$$

In contrast, to create an atlas for the orbifold  $\mathcal{H}$  represented by the groupoid with only two disjoint circles in the arrow space, we set  $M_{13}, M_{14}$  and  $M_{23}$  to be identical to the module described in (4.3). However, for the last inclusion we define  $M_{24}$  with action given by

$$(4.4) \quad \begin{array}{ccc} \text{left multiply by } \omega_j & & \begin{array}{ccccc} & \curvearrowright & & & \\ a_{ji} & \longrightarrow & b_{ji} & \longrightarrow & c_{ji} \end{array} \\ & & \begin{array}{ccccc} & \curvearrowright & & & \\ a_{ji} & \longleftarrow & b_{ji} & \longleftarrow & c_{ji} \end{array} \\ \text{right multiply by } \omega_i & & \end{array}$$

It is easy to see that all required compatibilities hold here. Note that these atlases can actually be constructed from the groupoid description of Example 2.1; we will discuss this process in Section 6.

**4.3. Further Results.** This section lists some results that generalize properties of the concrete embeddings to the abstract embeddings in an orbifold atlas.

**Lemma 4.16.** *For  $\lambda \in \text{Abst}(\mu_{kj})$  and  $\mu, \mu' \in \text{Abst}(\mu_{ji})$ , if  $\lambda \otimes \mu = \lambda \otimes \mu'$  then  $\mu = \mu'$ .*

*Proof.* If  $\lambda \otimes \mu = \lambda \otimes \mu'$  then  $\alpha_{kji}(\lambda \otimes \mu) = \alpha_{kji}(\lambda \otimes \mu')$  and hence,  $\tilde{\lambda} \circ \tilde{\mu} = \tilde{\lambda} \circ \tilde{\mu}'$  by Remark 4.13(1). Since  $\tilde{\lambda}$  is injective, this implies that  $\tilde{\mu} = \tilde{\mu}'$ , i.e.,  $\tilde{\rho}_{ji}(\mu) = \tilde{\rho}_{ji}(\mu')$ . By Remark 4.13(3), there is an element  $g \in G_i$  such that  $\mu \cdot g = \mu'$ . Hence,  $\lambda \otimes \mu = \lambda \otimes \mu' = \lambda \otimes (\mu \cdot g) = (\lambda \otimes \mu) \cdot g$ . However,  $G_i$  acts freely on  $\text{Abst}(\mu_{ji})$  by condition (2) of Definition 4.8. Hence,  $g = e_i$  and  $\mu = \mu'$ .  $\square$

**Lemma 4.17.** *For  $\lambda, \lambda' \in \text{Abst}(\mu_{kj})$  and  $\mu \in \text{Abst}(\mu_{ji})$ , if  $\lambda \otimes \mu = \lambda' \otimes \mu$  then  $\lambda = \lambda'$ .*

*Proof.* Since  $G_k$  acts transitively on  $\text{Abst}(\mu_{kj})$ , there is an element  $g \in G_k$  such that  $g \cdot \lambda = \lambda'$ . This implies that  $g \cdot \alpha_{kji}(\lambda \otimes \mu) = \alpha_{kji}(\lambda' \otimes \mu)$ . Since the action of  $G_k$  on  $\text{Abst}(\mu_{ki})$  is free, this implies that  $g = e_k$  and hence,  $\lambda = \lambda'$ .  $\square$

**Lemma 4.18.** *If  $U_i \leq U_j \leq U_k$  in  $\mathcal{O}(\mathcal{U})$  and  $\lambda \in \text{Abst}(\mu_{ki})$ ,  $\nu \in \text{Abst}(\mu_{kj})$  with  $\tilde{\lambda}(\tilde{U}_i) \cap \tilde{\nu}(\tilde{U}_j) \neq \emptyset$ , then there is a unique  $\kappa \in \text{Abst}(\mu_{ji})$  such that  $\alpha_{kji}(\nu \otimes \kappa) = \lambda$ .*

*Proof.* First note that by Lemma 4.2 there is a unique element  $\theta \in \text{Con}(\mu_{ji})$  such that  $\tilde{\nu} \circ \theta = \tilde{\lambda}$ . By Remark 4.13(2), there is an element  $\kappa' \in \text{Abst}(\mu_{ji})$  such that  $\tilde{\kappa}' = \theta$ . Then by Remark 4.13(1) we have  $\tilde{\rho}_{ji}(\alpha_{kji}(\nu \otimes \kappa')) = \tilde{\nu} \circ \tilde{\kappa}' = \tilde{\nu} \circ \theta = \tilde{\lambda} = \tilde{\rho}_{ji}(\lambda)$ . So using Remark 4.13(3) there is an element  $g \in G_i$  such that  $\alpha_{kji}(\nu \otimes (\kappa' \cdot g)) = \alpha_{kji}(\nu \otimes \kappa') \cdot g = \lambda$ . Let  $\kappa = \kappa' \cdot g$ . The fact that  $\kappa$  is unique with this property follows from Lemma 4.16 together with the fact that  $\alpha_{kji}$  is an isomorphism by condition (2) of Definition 4.8.  $\square$

The strong local compatibility property for concrete embeddings of atlas charts extends to abstract embeddings in the following way.

**Lemma 4.19.** *For any charts  $\tilde{U}_1, \tilde{U}_2, \tilde{U}_3$  with  $U_1 \leq U_3$  and  $U_2 \leq U_3$  in  $\mathcal{O}(\mathcal{U})$  with points  $x_i \in \tilde{U}_i$  for  $i = 1, 2, 3$  and abstract embeddings  $\lambda_{3i} \in \text{Abst}(\mu_{3i})$  such that  $\tilde{\lambda}_{3i}(x_i) = x_3$  for  $i = 1, 2$ , there is a chart  $\tilde{U}_4$  with a point  $y \in \tilde{U}_4$  and abstract embeddings  $\kappa_i \in \text{Abst}(\mu_{i4})$  such that  $\tilde{\kappa}_{i4}(y) = x_i$  for  $i = 1, 2$  and  $\alpha_{314}(\lambda_{31} \otimes \kappa_{14}) = \alpha_{324}(\lambda_{32} \otimes \kappa_{24})$ .*



*Proof.* By Lemma 4.3 there is a chart  $\tilde{U}_4$  in the atlas  $\mathcal{U}$  with  $U_4 \subseteq U_1 \cap U_2$ , and concrete embeddings  $\nu_{i4} \in \text{Con}(\mu_{4i})$  for  $i = 1, 2$ , such that  $\tilde{\lambda}_{31} \circ \nu_{14} = \tilde{\lambda}_{32} \circ \nu_{24}$ . Furthermore, there is a point  $y \in \tilde{U}_4$ , such that  $\nu_{i4}(y) = x_i$  for  $i = 1, 2$ . Since  $\tilde{\rho}_{14}$  and  $\tilde{\rho}_{24}$  are surjective by Remark 4.13(2), there are abstract embeddings  $\theta_{i4}$  with  $\tilde{\theta}_{i4} = \nu_{i4}$  for  $i = 1, 2$ . Then

$$\begin{aligned} \tilde{\rho}_{34}(\alpha_{314}(\lambda_{31} \otimes \theta_{14})) &= \tilde{\lambda}_{31} \circ \tilde{\theta}_{14} \\ &= \tilde{\lambda}_{31} \circ \nu_{14} \\ &= \tilde{\lambda}_{32} \circ \nu_{24} \\ &= \tilde{\lambda}_{32} \circ \tilde{\theta}_{24} \\ &= \tilde{\rho}_{34}(\alpha_{324}(\lambda_{32} \otimes \theta_{24})). \end{aligned}$$

By Remark 4.13(3) there is an element  $g \in G_4$  such that  $\alpha_{314}(\lambda_{31} \otimes \theta_{14}) \cdot g = \alpha_{324}(\lambda_{32} \otimes \theta_{24})$  and hence,  $\alpha_{314}(\lambda_{31} \otimes \theta_{14} \cdot g) = \alpha_{324}(\lambda_{32} \otimes \theta_{24})$ .

If we define  $\kappa_{14} = \theta_{14} \cdot g$  and  $\kappa_{24} = \theta_{24}$ , then we immediately have that  $\tilde{\kappa}_{24}(y) = x_2$  and  $\alpha_{314}(\lambda_{31} \otimes \kappa_{14}) = \alpha_{324}(\lambda_{32} \otimes \kappa_{24})$ . These properties imply that

$$\begin{aligned} \tilde{\lambda}_{31} \circ \tilde{\kappa}_{14}(y) &= \tilde{\lambda}_{32} \circ \tilde{\kappa}_{24}(y) \\ &= \tilde{\lambda}_{32}(x_2) \\ &= x_3 \\ &= \tilde{\lambda}_{31}(x_1). \end{aligned}$$

Since  $\tilde{\lambda}_{31}$  is injective,  $\tilde{\kappa}_{14}(y) = x_1$ . So  $\kappa_{14}$  and  $\kappa_{24}$  have the required property.  $\square$

## 5. CONSTRUCTING A GROUPOID FROM AN ORBIFOLD ATLAS

Let  $\mathfrak{U}$  be an orbifold atlas as in Definition 4.8; to fix notation, say that we have open sets  $U_i$  with inclusions given by  $\mu_{ji} : U_i \rightarrow U_j$ , and atlas bimodules  $\text{Abst}(\mu_{ji}) : G_i \rightharpoonup G_j$ , with pseudofunctor structure defined by identity isomorphisms  $\alpha_i : G_i \Rightarrow \text{Abst}(\mu_{ii})$  for each  $U_i$  in  $\mathcal{O}(\mathfrak{U})$  and composition isomorphisms  $\alpha_{kji} : \text{Abst}(\mu_{kj}) \otimes \text{Abst}(\mu_{ji}) \Rightarrow \text{Abst}(\mu_{ki})$  for each composable pair of arrows  $U_i \xrightarrow{\mu_{ji}} U_j \xrightarrow{\mu_{kj}} U_k$  in  $\mathcal{O}(\mathfrak{U})$ . In order to obtain an étale groupoid representation of this orbifold, we will first construct a smooth category  $\mathcal{C}(\mathfrak{U})$  for which the arrows do not necessarily have inverses, and then construct its smooth groupoid of fractions by an internal version of the Gabriel-Zisman construction as described in [PSi].

The smooth category  $\mathcal{C}(\mathfrak{U})$  has space of objects defined by the disjoint union of the charts:

$$\mathcal{C}(\mathfrak{U})_0 = \coprod_{\tilde{U}_i \in \mathcal{U}} \tilde{U}_i.$$

The space of arrows  $\mathcal{C}(\mathfrak{U})_1$  is constructed using the atlas bimodules  $\text{Abst}$ , where each  $\text{Abst}(\mu_{ji})$  has the discrete topology:

$$\mathcal{C}(\mathfrak{U})_1 = \coprod_{\mu_{ji} \in \mathcal{O}(\mathfrak{U})} \text{Abst}(\mu_{ji}) \times \tilde{U}_i.$$

The source map is defined by the projection  $s(\lambda, x) = x \in \tilde{U}_i$ , and the target map uses the concrete embedding  $\tilde{\lambda}$  described in Lemma 4.12,  $t(\lambda, x) = \tilde{\lambda}(x) \in \tilde{U}_j$ . The unit map  $u: \mathcal{C}(\mathfrak{U})_0 \rightarrow \mathcal{C}(\mathfrak{U})_1$  is given by  $u(x) = (\alpha_i(e_i), x)$  for  $x \in \tilde{U}_i$ .

We define the composition  $m: \mathcal{C}(\mathfrak{U})_1 \times_{\mathcal{C}(\mathfrak{U})_0} \mathcal{C}(\mathfrak{U})_1 \rightarrow \mathcal{C}(\mathfrak{U})_1$  in  $\mathcal{C}(\mathfrak{U})$  using the composition isomorphisms  $\alpha_{kji}$  of **Abst**: Suppose that  $(\lambda, x) \in \mathbf{Abst}(\mu_{ji}) \times \tilde{U}_i$  and  $(\lambda', x') \in \mathbf{Abst}(\mu_{kj}) \times \tilde{U}_j$ , such that  $t(\lambda, x) = s(\lambda', x')$ ; then we know that  $x' = \tilde{\lambda}(x)$  and we define

$$m((\lambda', x'), (\lambda, x)) = (\alpha_{kji}(\lambda' \otimes \lambda), x).$$

This gives us a smooth category with structure maps that are étale, since they are embeddings when restricted to any connected component of the space of arrows.

Next, we want to construct a smooth groupoid  $\mathcal{G}(\mathfrak{U})$  from the smooth category  $\mathcal{C}(\mathfrak{U})$ . We do this using a category of fractions construction. For ordinary categories, the groupoid of fractions can be constructed using the Gabriel-Zisman span construction whenever the following two conditions hold:

- (O) *The Ore condition*: for each cospan of arrows  $A \xrightarrow{f} C \xleftarrow{w} B$  there are arrows to complete this to a commutative square

$$\begin{array}{ccc} D & \xrightarrow{g} & B \\ v \downarrow & & \downarrow w \\ A & \xrightarrow{f} & C \end{array}$$

- (WC) *Weak cancellability*: for any arrows  $C \xrightleftharpoons[g]{f} B \xrightarrow{h} A$  for which  $hf = hg$ , there is an arrow  $j: D \rightarrow C$  such that  $fj = gj$ .

In the Gabriel-Zisman groupoid of fractions, each arrow is represented by a span

$$\begin{array}{ccc} \xleftarrow{f} & \xrightarrow{g} & \\ \xleftarrow{f_2} & \xrightarrow{g_2} & \end{array} \quad \text{in the original category, and two such spans, } \begin{array}{ccc} \xleftarrow{f_1} & \xrightarrow{g_1} & \\ \xleftarrow{f_2} & \xrightarrow{g_2} & \end{array} \quad \text{and} \quad \begin{array}{ccc} \xleftarrow{f_2} & \xrightarrow{g_2} & \\ \xleftarrow{f_1} & \xrightarrow{g_1} & \end{array}$$

represent the same arrow when there is a third span  $\begin{array}{ccc} \xleftarrow{h_1} & \xrightarrow{h_2} & \end{array}$  to make the following diagram commute,

$$\begin{array}{ccccc} & & & & \\ & f_1 & & g_1 & \\ & \swarrow & \uparrow h_1 & \searrow & \\ & f_2 & & g_2 & \\ & \swarrow & \downarrow h_2 & \searrow & \end{array}$$

(5.1)

This relation is an equivalence relation and defines a congruence on the arrow structure when the category satisfies the conditions (O) and (WC).

In our case, we want to ensure that the resulting groupoid carries an induced topological structure. Hence, we want to use the analogous construction internal to the category of topological spaces as described in [PSi]. The construction described there starts with a topological category and gives topological versions of the (O) and (WC) conditions. It then considers the space of spans of arrows in  $\mathcal{C}(\mathfrak{U})$ , constructed as the pullback  $\mathcal{C}(\mathfrak{U})_1 \times_{\mathcal{C}(\mathfrak{U})_0} \mathcal{C}(\mathfrak{U})_1$  of  $s$  along  $s$ , and a space encoding diagrams of the form (5.1). This latter space has two obvious projection maps to  $\mathcal{C}(\mathfrak{U})_1 \times_{\mathcal{C}(\mathfrak{U})_0} \mathcal{C}(\mathfrak{U})_1$ . Then  $\mathcal{G}(\mathfrak{U})_1$  is obtained as the coequalizer of these projection

maps, and they form its kernel pair. On the point level this still implies that the relation described here is an equivalence relation on the space of spans.

We also want the resulting groupoid  $\mathcal{G}(\mathfrak{U})$  to inherit the smooth structure of  $\mathcal{C}(\mathfrak{U})$ . However, working out the corresponding conditions inside the category of smooth manifolds is a bit tricky, as not all pullbacks exist there. Fortunately, we can work with the topological version; we will show that the spaces we obtain have induced smooth structures that make all structure maps local diffeomorphisms.

Rather than reviewing the general construction from [PSi] in detail, we will just focus on what it means for our atlas category  $\mathcal{C}(\mathfrak{U})$ . The condition corresponding to the (O) condition above requires us to consider the spaces of cospans of arrows and of commutative squares in  $\mathcal{C}(\mathfrak{U})$ .

Since the space of arrows  $\mathcal{C}(\mathfrak{U})$  is defined by the disjoint union of charts, the space  $\text{Cospan}(\mathcal{C}(\mathfrak{U})) = \mathcal{C}(\mathfrak{U})_1 \times_{t, \mathcal{C}(\mathfrak{U})_0, t} \mathcal{C}(\mathfrak{U})_1$  representing the cospans of arrows is a coproduct of pullbacks: for each pair  $\lambda \in \text{Abst}(\mu_{ki})$  and  $\xi \in \text{Abst}(\mu_{kj})$ , we create the pullback

$$\begin{array}{ccc} P(\lambda, \xi) & \longrightarrow & \tilde{U}_j \\ \downarrow & & \downarrow \tilde{\xi} \\ \tilde{U}_i & \xrightarrow{\tilde{\lambda}} & \tilde{U}_k \end{array}$$

Then

$$\text{Cospan}(\mathcal{C}(\mathfrak{U})) = \coprod_{\substack{\mu_{ki}, \mu_{kj} \in \mathcal{C}(\mathfrak{U}) \\ \lambda \in \text{Abst}(\mu_{ki}) \\ \xi \in \text{Abst}(\mu_{kj})}} P(\lambda, \xi).$$

Note that since both  $\tilde{\lambda}$  and  $\tilde{\xi}$  are open embeddings, the space  $P(\lambda, \xi)$  is homeomorphic to  $\tilde{\lambda}(\tilde{U}_i) \cap \tilde{\xi}(\tilde{U}_j) \subseteq \tilde{U}_k$  and hence carries a canonical smooth structure.

In the general theory of internal categories of fractions, the space  $\text{ComSq}(\mathcal{C}(\mathfrak{U}))$  representing commutative squares is obtained as an equalizer of composition maps from the space of all possible squares to  $\mathcal{C}(\mathfrak{U})_1$ . In this case, the space encoding all possible squares is a large coproduct of charts,

$$\coprod \tilde{U}_\ell,$$

taken over all combinations of  $(\lambda, \gamma, \xi, \delta) \in \text{Abst}(\mu_{ki}) \times \text{Abst}(\mu_{i\ell}) \times \text{Abst}(\mu_{kj}) \times \text{Abst}(\mu_{j\ell})$ . The two composition maps send a point  $(x, \lambda, \gamma, \xi, \delta)$  to  $(x, \alpha_{ki\ell}(\lambda \otimes \gamma))$  and  $(x, \alpha_{kj\ell}(\xi \otimes \delta))$  respectively. So we see that the space of commutative squares is a coproduct of equalizers. Moreover, in our situation the two parallel embeddings  $\alpha_{ki\ell}(\lambda \otimes \gamma)$  and  $\alpha_{kj\ell}(\xi \otimes \delta)$  either agree everywhere or nowhere, and hence the space becomes simply a coproduct of charts:

$$\text{ComSq}(\mathcal{C}(\mathfrak{U})) = \coprod \tilde{U}_\ell$$

where the coproduct is taken over all  $(\lambda, \gamma, \xi, \delta) \in \text{Abst}(\mu_{ki}) \times \text{Abst}(\mu_{i\ell}) \times \text{Abst}(\mu_{kj}) \times \text{Abst}(\mu_{j\ell})$  such that  $\alpha_{ki\ell}(\lambda \otimes \gamma) = \alpha_{kj\ell}(\xi \otimes \delta)$ .

There is a projection map  $\varphi: \text{ComSq}(\mathcal{C}(\mathfrak{U})) \rightarrow \text{Cospan}(\mathcal{C}(\mathfrak{U}))$  given by

$$\varphi(x, \lambda, \gamma, \xi, \delta) = (\lambda, \tilde{\gamma}(x), \tilde{\delta}(x), \xi).$$

$$(x, \lambda_1, \lambda_2, \lambda_4) \mapsto (\tilde{\lambda}_1(x), \lambda_2, \lambda_4).$$

Since this is an embedding when restricted to any connected component, it is clear that this is a local homeomorphism (even a local diffeomorphism). To see that it is surjective, given  $(z, \lambda_2, \lambda_4)$  where  $z \in \tilde{U}_j$ ,  $\lambda_2 \in \mathbf{Abst}(\mu_{kj})$  and  $\lambda_4 \in \mathbf{Abst}(\mu_{\ell k})$ , we can define  $U_i = U_j$ ,  $x = z$ , and  $\lambda_1 \in \mathbf{Abst}(\mu_{ii})$  such that  $\tilde{\lambda}_1$  is the identity on  $\tilde{U}_i$ ; this gives an object  $(x, \lambda_1, \lambda_2, \lambda_4)$  such that  $\mathbf{m}(x, \lambda_1, \lambda_2, \lambda_4) = (z, \lambda_2, \lambda_4)$ .  $\square$

Thus, [PSi] says that we have the conditions necessary to construct a topological groupoid of fractions. This construction gives us the following atlas groupoid  $\mathcal{G}(\mathfrak{U})$ . The space of objects remains the same:

$$\mathcal{G}(\mathfrak{U})_0 = \coprod_{\tilde{U}_i \in \mathcal{U}} \tilde{U}_i.$$

To obtain the space of arrows  $\mathcal{G}(\mathfrak{U})_1$ , we start with the space of spans

$$\text{Span}(\mathcal{C}(\mathfrak{U})) = \coprod_{\mu_{ji}, \mu_{ki} \in \mathcal{O}(\mathcal{U})} \mathbf{Abst}(\mu_{ji}) \times \tilde{U}_i \times \mathbf{Abst}(\mu_{ki}).$$

We define  $\mathcal{G}(\mathfrak{U})_1$  as a coequalizer encoding the diagrams of the form (5.1), which in this case is the quotient of the space  $\text{Span}(\mathcal{C}(\mathfrak{U}))$  by the equivalence relation  $\mathcal{R}_{\mathfrak{U}}$ , described as follows: Suppose  $U_i, U_{i'} \leq U_j$ , and  $U_i, U_{i'} \leq U_k$  in  $\mathcal{O}(\mathcal{U})$ , and fix  $(\lambda, x, \nu), (\lambda', x', \nu')$  for  $x \in \tilde{U}_i$ ,  $x' \in \tilde{U}_{i'}$ ,  $\lambda \in \mathbf{Abst}(\mu_{ji})$ ,  $\lambda' \in \mathbf{Abst}(\mu_{ji'})$ ,  $\nu \in \mathbf{Abst}(\mu_{ki})$ ,  $\nu' \in \mathbf{Abst}(\mu_{ki'})$ . Then,

$$((\lambda, x, \nu), (\lambda', x', \nu')) \in \mathcal{R}_{\mathfrak{U}}$$

if and only if there is a chart  $\tilde{U}_{\ell}$  with a point  $y \in \tilde{U}_{\ell}$  and elements  $\kappa \in \mathbf{Abst}(\mu_{i\ell})$  and  $\kappa' \in \mathbf{Abst}(\mu_{i'\ell})$  such that

$$\tilde{\kappa}(y) = x, \quad \tilde{\kappa}'(y) = x', \quad \alpha_{ji\ell}(\lambda \otimes \kappa) = \alpha_{ji'\ell}(\lambda' \otimes \kappa') \text{ and } \alpha_{ki\ell}(\nu \otimes \kappa) = \alpha_{ki'\ell}(\nu' \otimes \kappa').$$

The space of arrows of the groupoid is then

$$\mathcal{G}(\mathfrak{U})_1 = \text{Span}(\mathcal{C}(\mathfrak{U}))/\mathcal{R}_{\mathfrak{U}}.$$

The structure maps of the groupoid are defined as follows. For  $x \in \tilde{U}_i$ ,  $\lambda \in \mathbf{Abst}(\mu_{ji})$  and  $\nu \in \mathbf{Abst}(\mu_{ki})$ :

$$\begin{aligned} s[\lambda, x, \nu] &= \tilde{\lambda}(x) \\ t[\lambda, x, \nu] &= \tilde{\nu}(x) \\ u(x) &= [\alpha_i(e_i), x, \alpha_i(e_i)] \end{aligned}$$

where  $e_i$  is the identity of  $G_i$ . By Remark 4.13(1), the maps  $s$  and  $t$  are well defined on equivalence classes.

We also need to define the composition operation on the arrow space of this groupoid. Suppose we have composable elements  $[\lambda_1, x_1, \nu_1]$  and  $[\lambda_2, x_2, \nu_2]$  with  $\lambda_1 \in \mathbf{Abst}(\mu_{j_1 i_1})$ ,  $\nu_1 \in \mathbf{Abst}(\mu_{j_0 i_1})$ ,  $\lambda_2 \in \mathbf{Abst}(\mu_{j_0 i_2})$ ,  $\nu_2 \in \mathbf{Abst}(\mu_{j_2 i_2})$ ; so we know that  $\tilde{\nu}_1(x_1) = \tilde{\lambda}_2(x_2)$ . By Lemma 4.19 there is a chart  $\tilde{U}_k$  with a point  $y \in \tilde{U}_k$  and abstract embeddings  $\kappa_n \in \mathbf{Abst}(\mu_{i_n k})$  for  $n = 1, 2$  such that  $\tilde{\kappa}_n(y) = x_n$  for  $n = 1, 2$  and  $\alpha_{j_0 i_1 k}(\nu_1 \otimes \kappa_1) = \alpha_{j_0 i_2 k}(\lambda_2 \otimes \kappa_2)$ . Then we define

$$m([\lambda_2, x_2, \nu_2], [\lambda_1, x_1, \nu_1]) = [\alpha_{j_1 i_1 k}(\lambda_1 \otimes \kappa_1), y, \alpha_{j_2 i_2 k}(\nu_2 \otimes \kappa_2)].$$

The fact that this is well defined on equivalence classes was proved in [PSi].

Lastly, the inverse of an arrow in  $\mathcal{G}(\mathfrak{U})$  is given by

$$i[\lambda, x, \nu] = [\nu, x, \lambda].$$

This is the topological groupoid produced by the construction of [PSi].

We now want to show that this groupoid represents an orbifold; that is, it has the required topological properties. Specifically, we show that the arrow space is Hausdorff, that there is a natural smooth structure such that all structure maps are local diffeomorphisms, and that the diagonal  $(s, t): \mathcal{G}(\mathfrak{U})_1 \rightarrow \mathcal{G}(\mathfrak{U})_0 \times \mathcal{G}(\mathfrak{U})_0$  is a proper map.

**Proposition 5.3.** *The space of arrows  $\mathcal{G}(\mathfrak{U})_1$  is Hausdorff.*

*Proof.* We need to show that the quotient  $\mathcal{G}(\mathfrak{U})_1 = \text{Span}(\mathcal{C}(\mathfrak{U}))/\mathcal{R}_{\mathfrak{U}}$  is Hausdorff. The original space of arrows  $\mathcal{C}(\mathfrak{U})_1$ , defined by a disjoint union of charts, is clearly Hausdorff, as is  $\text{Span}(\mathcal{C}(\mathfrak{U}))$ . Therefore it suffices to show that the subspace

$$\mathcal{R}_{\mathfrak{U}} = \{((\lambda, x, \nu), (\lambda', x', \nu'))\}$$

defined by the equivalence relation above is closed in  $\text{Span}(\mathcal{C}(\mathfrak{U})) \times \text{Span}(\mathcal{C}(\mathfrak{U}))$ .

We show that the complement is open. So suppose that  $((\lambda, x, \nu), (\lambda', x', \nu')) \in (\text{Abst}(\mu_{ji}) \times \tilde{U}_i \times \text{Abst}(\mu_{ki})) \times (\text{Abst}(\mu_{j'i'}) \times \tilde{U}_{i'} \times \text{Abst}(\mu_{k'i'})) \subset (\text{Span}(\mathcal{C}(\mathfrak{U})) \times \text{Span}(\mathcal{C}(\mathfrak{U}))) \setminus \mathcal{R}_{\mathfrak{U}}$ , so that  $(\lambda, x, \nu)$  and  $(\lambda', x', \nu')$  are not related under  $\mathcal{R}_{\mathfrak{U}}$ . We produce an open set  $\mathcal{O}$  containing  $((\lambda, x, \nu), (\lambda', x', \nu'))$  such that  $\mathcal{R}_{\mathfrak{U}} \cap \mathcal{O} = \emptyset$ .

If  $j' \neq j$  or  $k' \neq k$ , then we can define  $\mathcal{O} = \left(\{\lambda\} \times \tilde{U}_i \times \{\nu\}\right) \times \left(\{\lambda'\} \times \tilde{U}_{i'} \times \{\nu'\}\right)$ .

If  $j = j'$  and  $k = k'$  but  $\tilde{\lambda}(x) \neq \tilde{\lambda}'(x')$ , we can use the Hausdorff property of  $\tilde{U}_j$  to find disjoint open neighbourhoods  $\tilde{U}_{\tilde{\lambda}(x)}$  and  $\tilde{U}_{\tilde{\lambda}'(x')}$ , and take their preimages under  $\tilde{\lambda}$  and  $\tilde{\lambda}'$  to obtain open neighbourhoods  $\tilde{U}_x$  of  $x$  and  $\tilde{U}_{x'}$  of  $x'$ . These give us open neighbourhoods  $\{\lambda\} \times \tilde{U}_x \times \{\nu\} \subseteq \{\lambda\} \times \tilde{U}_i \times \{\nu\}$  and  $\{\lambda'\} \times \tilde{U}_{x'} \times \{\nu'\} \subseteq \{\lambda'\} \times \tilde{U}_{i'} \times \{\nu'\}$  of  $(\lambda, x, \nu)$  and  $(\lambda', x', \nu')$  respectively in  $\text{Span}(\mathcal{C}(\mathfrak{U}))$ , and we define  $\mathcal{O} = \left(\{\lambda\} \times \tilde{U}_x \times \{\nu\}\right) \times \left(\{\lambda'\} \times \tilde{U}_{x'} \times \{\nu'\}\right)$ . A similar argument produces  $\mathcal{O}$  when  $\tilde{\nu}(x) \neq \tilde{\nu}'(x')$ .

Lastly, we consider the case where  $j = j'$ ,  $k = k'$ ,  $\tilde{\lambda}(x) = \tilde{\lambda}'(x')$  and  $\tilde{\nu}(x) = \tilde{\nu}'(x')$ , but  $(\lambda, x, \nu)$  and  $(\lambda', x', \nu')$  are not related under  $\mathcal{R}_{\mathfrak{U}}$ . By Lemma 4.19 there is a chart  $\tilde{U}_\ell$  with a point  $y$  and a pair of abstract embeddings  $\kappa \in \text{Abst}(\mu_{i\ell})$  and  $\kappa' \in \text{Abst}(\mu_{i'\ell})$ , such that  $\tilde{\kappa}(y) = x$ ,  $\tilde{\kappa}'(y) = x'$  and  $\alpha_{j i \ell}(\lambda \otimes \kappa) = \alpha_{j i' \ell}(\lambda' \otimes \kappa')$ . Since  $(\lambda, x, \nu)$  and  $(\lambda', x', \nu')$  are not related by  $\mathcal{R}_{\mathfrak{U}}$ , we know that

$$(5.4) \quad \alpha_{k i \ell}(\nu \otimes \kappa) \neq \alpha_{k i' \ell}(\nu' \otimes \kappa').$$

Now we consider the open neighbourhoods  $\{\lambda\} \times \tilde{\kappa}(\tilde{U}_\ell) \times \{\nu\}$  of  $(\lambda, x, \nu)$  and  $\{\lambda'\} \times \tilde{\kappa}'(\tilde{U}_\ell) \times \{\nu'\}$  of  $(\lambda', x', \nu')$ , and define

$$\mathcal{O} = \left(\{\lambda\} \times \tilde{\kappa}(\tilde{U}_\ell) \times \{\nu\}\right) \times \left(\{\lambda'\} \times \tilde{\kappa}'(\tilde{U}_\ell) \times \{\nu'\}\right).$$

We claim that  $\mathcal{R}_{\mathfrak{U}} \cap \mathcal{O} = \emptyset$ : suppose by contradiction that there are points  $z \in \tilde{\kappa}(\tilde{U}_\ell)$  and  $z' \in \tilde{\kappa}'(\tilde{U}_\ell)$  such that  $(\lambda, z, \nu) \sim (\lambda', z', \nu')$ . Then there is a chart  $\tilde{U}_m$  with a point  $u \in \tilde{U}_m$  and abstract embeddings  $\theta \in \text{Abst}(\mu_{im})$  and  $\theta' \in \text{Abst}(\mu_{i'm})$  such that  $\tilde{\theta}(u) = z$ ,  $\tilde{\theta}'(u) = z'$ ,  $\alpha_{j i m}(\lambda \otimes \theta) = \alpha_{j i' m}(\lambda' \otimes \theta')$  and  $\alpha_{k i m}(\nu \otimes \theta) = \alpha_{k i' m}(\nu' \otimes \theta')$ . Without loss of generality, we may assume that  $U_m \leq U_\ell$  in  $\mathcal{O}(\mathfrak{U})$ .

Since  $\tilde{\theta}(u) = z \in \tilde{\kappa}(\tilde{U}_\ell)$ , then by Lemma 4.18 there is a unique  $\zeta \in \mathbf{Abst}(\mu_{\ell m})$  such that  $\alpha_{i\ell m}(\kappa \otimes \zeta) = \theta$ . Now,

$$\begin{aligned} \alpha_{ji'm}(\lambda' \otimes \alpha_{i'\ell m}(\kappa' \otimes \zeta)) &= \alpha_{j\ell m}(\alpha_{ji'\ell}(\lambda' \otimes \kappa') \otimes \zeta) \\ &= \alpha_{j\ell m}(\alpha_{ji\ell}(\lambda \otimes \kappa) \otimes \zeta) \\ &= \alpha_{jim}(\lambda \otimes \alpha_{i\ell m}(\kappa \otimes \zeta)) \\ &= \alpha_{jim}(\lambda \otimes \theta) \\ &= \alpha_{ji'm}(\lambda' \otimes \theta'). \end{aligned}$$

Since  $\alpha_{ji'm}$  is an isomorphism, using the previous identity and Lemma 4.16, we get that  $\alpha_{i'\ell m}(\kappa' \otimes \zeta) = \theta'$ . Hence,

$$\begin{aligned} \alpha_{k\ell m}(\alpha_{kil}(\nu \otimes \kappa) \otimes \zeta) &= \alpha_{kim}(\nu \otimes \alpha_{i\ell m}(\kappa \otimes \zeta)) \\ &= \alpha_{kim}(\nu \otimes \theta) \\ &= \alpha_{ki'm}(\nu' \otimes \theta') \\ &= \alpha_{ki'm}(\nu' \otimes \alpha_{i'\ell m}(\kappa' \otimes \zeta)) \\ &= \alpha_{k\ell m}(\alpha_{ki'\ell}(\nu' \otimes \kappa') \otimes \zeta) \end{aligned}$$

By Lemma 4.17, this implies that  $\alpha_{kil}(\nu \otimes \kappa) = \alpha_{ki'\ell}(\nu' \otimes \kappa')$ , contradicting (5.4). This completes the proof of the claim and the proof of the proposition.  $\square$

**Lemma 5.4.** *The source map  $s$  of the groupoid  $\mathcal{G}(\mathfrak{U})_1$  is a local homeomorphism.*

*Proof.* By Lemma 4.16, two points  $(\lambda, x, \nu)$  and  $(\lambda, x', \nu)$  in  $\text{Span}(\mathcal{C}(\mathfrak{U}))$  with  $x, x' \in \tilde{U}$ ,  $\lambda \in \mathbf{Abst}(\mu_{VU})$  and  $\nu \in \mathbf{Abst}(\mu_{WU})$  are identified in  $\mathcal{G}(\mathfrak{U})_1$  if and only if  $x = x'$ . Hence the points of the form  $[\lambda, y, \nu]$  in  $\mathcal{G}(\mathfrak{U})_1$  (for  $y \in \tilde{U}$ ) form an open neighbourhood of  $[\lambda, x, \nu]$  which is homeomorphic to  $\tilde{U}$ , and  $s$  is a homeomorphism onto its image when restricted to this neighbourhood.  $\square$

Note that the space  $\text{Span}(\mathcal{C}(\mathfrak{U}))$  is a manifold and the quotient map  $\text{Span}(\mathcal{C}(\mathfrak{U})) \rightarrow \mathcal{G}(\mathfrak{U})_1$  is a local homeomorphism. In fact, it is a homeomorphism/embedding when restricted to each connected component, so  $\mathcal{G}(\mathfrak{U})_1$  inherits the structure of being a smooth manifold and the structure maps are local diffeomorphisms with respect to this structure.

**Proposition 5.5.** *The diagonal  $(s, t): \mathcal{G}(\mathfrak{U})_1 \rightarrow \mathcal{G}(\mathfrak{U})_0 \times \mathcal{G}(\mathfrak{U})_0$  is proper.*

*Proof.* We begin by showing that the fibers of  $(s, t)$  are finite, hence compact. Since  $\mathcal{G}(\mathfrak{U})$  is a groupoid, it is sufficient to consider the fibers of the form  $(s, t)^{-1}(x, x)$ . So let  $x \in \mathcal{G}(\mathfrak{U})_0$ , thus  $x \in \tilde{U}_i$  for some  $i$ , and consider  $(s, t)^{-1}(x, x)$ . First consider the elements in this fiber of the form  $[\lambda, x', \lambda']$  with  $\lambda, \lambda' \in \mathbf{Abst}(\mu_{ii})$  (which is isomorphic to  $G_i$  via  $\alpha_i^{-1}$ ) and  $x' \in \tilde{U}_i$ , such that  $\tilde{\lambda}(x') = x = \tilde{\lambda}'(x')$ . For any  $g \in G_i$ , we know that  $[\lambda, x', \lambda'] = [\lambda \cdot g, (\rho_i(g^{-1}))(x'), \lambda' \cdot g]$ . Since the right action of  $G_i$  is transitive on  $\mathbf{Abst}(\mu_{ii})$ , each  $[\lambda, x', \lambda']$  has a representation for which  $\lambda = \alpha_i(e)$ , and consequently  $\tilde{\lambda} = \text{id}_{\tilde{U}_i}$ . So in this representation, we have  $[\lambda, x, \lambda']$ , with  $\lambda = \alpha_i(e)$  and  $\lambda'$  such that  $\tilde{\lambda}'(x) = x$ .

Fix one of these  $\lambda'$ . We claim that all the other arrows in  $(s, t)^{-1}(x, x)$  are of the form  $[\lambda, x, g \cdot \lambda']$  with  $g \in (G_i)_x = \{g \in G_i \text{ such that } \rho_i(g_i) \cdot x = x\}$ . To see that this is true, suppose that  $[\nu, y, \nu'] \in (s, t)^{-1}(x, x)$  with  $y \in \tilde{U}_j$  and  $\nu, \nu' \in \mathbf{Abst}(\mu_{ij})$ . If  $g \in G_i$  is the unique element such that  $g \cdot \alpha_{iij}(\lambda' \otimes \nu) = \nu'$ , then  $[\lambda, x, g \cdot \lambda'] = [\nu, y, \nu']$ . This implies that  $(s, t)^{-1}(x, x)$  is finite.

It remains to show that  $(s, t)$  is a closed map. We begin by showing that the image  $(s, t)(\mathcal{G}(\mathfrak{U})_1) \subseteq \mathcal{G}(\mathfrak{U})_0 \times \mathcal{G}(\mathfrak{U})_0$  is closed. Let  $(x, y) \in \mathcal{G}(\mathfrak{U})_0 \times \mathcal{G}(\mathfrak{U})_0$ , with  $x \in \tilde{U}_j$  and  $y \in \tilde{U}_k$ . If  $(x, y) \in (\mathcal{G}(\mathfrak{U})_0 \times \mathcal{G}(\mathfrak{U})_0) \setminus (s, t)(\mathcal{G}(\mathfrak{U})_1)$ , then the images  $\bar{x}$  and  $\bar{y}$  in the quotient space are distinct. Since the quotient space is Hausdorff, there are disjoint open neighbourhoods  $U_{\bar{x}}$  and  $U_{\bar{y}}$  of  $\bar{x}$  and  $\bar{y}$  in  $\mathcal{G}(\mathfrak{U})_0/\mathcal{G}(\mathfrak{U})_1$ . Define  $\tilde{U}_{\bar{x}}$  to be the preimage of  $U_{\bar{x}}$  in  $\tilde{U}_j$  under the quotient map, and similarly let  $\tilde{U}_{\bar{y}}$  be the preimage of  $U_{\bar{y}}$  in  $\tilde{U}_k$ . Then  $(\tilde{U}_{\bar{x}} \times \tilde{U}_{\bar{y}}) \cap (s, t)(\mathcal{G}(\mathfrak{U})_1) = \emptyset$ , and we have shown that the complement of  $(s, t)(\mathcal{G}(\mathfrak{U})_1)$  is open.

Now let  $C \subseteq \mathcal{G}(\mathfrak{U})_1$  be a closed subset, and let  $(x, y) \in (\mathcal{G}(\mathfrak{U})_0 \times \mathcal{G}(\mathfrak{U})_0) \setminus (s, t)(C)$ , where  $x \in \tilde{U}_j$  and  $y \in \tilde{U}_k$ . Without loss of generality we may assume that  $(s, t)^{-1}(x, y) \neq \emptyset$ . We have shown earlier that it is finite, and therefore we have  $(s, t)^{-1}(x, y) = \{[\lambda_1, z_1, \lambda'_1], \dots, [\lambda_n, z_n, \lambda'_n]\}$  with  $z_i \in \tilde{U}_{m_i}$  and  $\lambda_i \in \text{Abst}(\mu_{j_{m_i}})$ ,  $\lambda'_i \in \text{Abst}(\mu_{k_{m_i}})$  for each  $i = 1, \dots, n$ . Since  $C \subseteq \mathcal{G}(\mathfrak{U})_1$  is closed, for each  $i$  there is an open neighbourhood  $\tilde{U}'_{m_i}$  of  $z_i$  in  $\tilde{U}_{m_i}$ , such that the set  $V_i = \{\lambda_i\} \times \tilde{U}'_{m_i} \times \{\lambda'_i\}$  does not intersect  $C$ .

Both  $s$  and  $t$  are homeomorphisms when restricted to each of the  $V_i$ , so  $s(V_i) \times t(V_i)$  is open in  $\mathcal{G}(\mathfrak{U})_0 \times \mathcal{G}(\mathfrak{U})_0$ . Then  $W_x \times W_y := (\bigcap_{i=1}^n s(V_i)) \times (\bigcap_{i=1}^n t(V_i))$  is an open neighbourhood of the point  $(x, y)$ . However,  $(s, t)^{-1}(W_x \times W_y)$  may contain connected components that are different from the  $V_i$  chosen before and hence may have a nonempty intersection with  $C$ . In order to make sure that there are no such components we need to choose smaller connected neighbourhoods  $W'_x \subseteq W_x \subseteq \tilde{U}_j$  and  $W'_y \subseteq W_y \subseteq \tilde{U}_k$  such that the isotropy groups of all points in  $W'_x$  are subgroups of the isotropy group of  $x$  and similarly for  $W'_y$ . We show how to construct  $W'_x$ . ( $W'_y$  can be found similarly.) For each  $g \in G_j$ , consider  $\rho_j(g)(W_x)$ . If  $x \in \rho_j(g)(W_x)$ , let  $W_x^g = \rho_j(g)(W_x)$ . If  $x \notin \rho_j(g)(W_x)$ , note that  $x$  is also not in the closure  $\overline{\rho_j(g)(W_x)}$  and let  $W_x^g = W_x \setminus \overline{\rho_j(g)(W_x)}$ . Now define  $W'_x = \bigcap_{g \in G_j} W_x^g$ . Then  $(s, t)^{-1}(W'_x \times W'_y) \subseteq \bigcup V_i$  and hence  $W'_x \times W'_y$  has an empty intersection with  $(s, t)(C)$ .  $\square$

## 6. CONSTRUCTING AN ORBIFOLD ATLAS FROM A GROUPOID

Given a smooth étale groupoid  $\mathcal{G}$  with a proper diagonal, we will construct an orbifold atlas for its quotient space  $\mathcal{G}_0/\mathcal{G}_1$ . We begin by reviewing a couple of results about these groupoids from [MP].

Let  $s, t: \mathcal{G}_1 \rightarrow \mathcal{G}_0$  denote the source and target maps of this groupoid. Since  $s$  and  $t$  are étale, the preimage  $s^{-1}(x) \cap t^{-1}(x) = (s, t)^{-1}(x, x)$  is a discrete group for any point  $x \in \mathcal{G}_0$ ; and since  $(s, t): \mathcal{G}_1 \rightarrow \mathcal{G}_0 \times \mathcal{G}_0$  is proper, this pre-image is a finite group. We denote it by  $\mathcal{G}_x$ .

Each point  $g \in \mathcal{G}_1$  has a neighbourhood  $W_g$  such that both  $s$  and  $t$  restrict to homeomorphisms on  $W_g$ . Such neighbourhoods are called *local bisections*. When  $s: W_g \rightarrow \tilde{U}$  and  $t: W_g \rightarrow \tilde{V}$ , we also denote this local bisection by its section  $\sigma: \tilde{U} \rightarrow W_g \subseteq \mathcal{G}_1$ ; i.e.,  $s \circ \sigma = \text{id}_{\tilde{U}}$ . In this case we say that this local bisection carries  $\tilde{U}$  onto  $(t \circ \sigma)(\tilde{U}) = \tilde{V}$ . We will also consider the situation where we take  $\tilde{V}$  to strictly contain the image  $t(W_g)$ ; i.e., when  $(t \circ \sigma)(\tilde{U}) \subseteq \tilde{V}$ . In this case we will say that the local bisection takes or embeds  $\tilde{U}$  into  $\tilde{V}$ .

The properness of the diagonal implies the existence of a collection of special open neighbourhoods for points  $x \in \mathcal{G}_0$ , the space of objects. In developing these,



we will consider invariant sets  $\tilde{U}$  of  $\mathcal{G}_0$ , and we will denote its quotient  $\tilde{U}/\mathcal{G}_1$  in  $\mathcal{G}_0/\mathcal{G}_1$  by  $U$ .

**Definition 6.1.** A connected simply connected open subset  $\tilde{U} \subseteq \mathcal{G}_0$  is a *translation open set* if the group  $G_U$  of local bisections  $\sigma: \tilde{U} \rightarrow \mathcal{G}_1$  (i.e.,  $s \circ \sigma = \text{id}_{\tilde{U}}$ ) which are defined on  $\tilde{U}$  and carry  $\tilde{U}$  onto itself (i.e.,  $(t \circ \sigma)(\tilde{U}) = \tilde{U}$ ) is finite and the homomorphism of Lie groupoids  $G_U \times \tilde{U} \rightarrow \mathcal{G}|_{\tilde{U}}$ , defined by evaluation  $(\sigma, u) \mapsto \sigma(u)$ , is an isomorphism.

These translation neighbourhoods form a basis for the topology on  $\mathcal{G}_0$ , as do the local bisections for the topology on  $\mathcal{G}_1$ . (This was shown for translation neighbourhoods in the proof of the part  $4 \Rightarrow 1$  of the main theorem of [MP]. Although that paper is about effective orbifolds, this part of the proof does not use effectiveness.)

Translation neighbourhoods and local bisections will form the essential building blocks for the orbifold atlas. Before defining the atlas, we prove some basic results about the structure of these neighbourhoods.

**6.1. Translation Neighbourhoods and Local Bisections.** We start with a couple of results about the relationship between translation neighbourhoods and local bisections. For any two open subsets  $\tilde{U}, \tilde{V} \subseteq \mathcal{G}_0$ , let  $A_{VU}$  denote the set of all local bisections  $\sigma: \tilde{U} \rightarrow \mathcal{G}(U, V) = s^{-1}(\tilde{U}) \cap t^{-1}(\tilde{V})$  which are defined on  $\tilde{U}$  and carry  $\tilde{U}$  into  $\tilde{V}$  in the sense that  $(t \circ \sigma)(\tilde{U}) \subseteq \tilde{V}$ . Note that there is a canonical action of  $G_V$  on  $A_{VU}$  from the left, and canonical action of  $G_U$  on  $A_{VU}$  from the right. Both actions arise from the standard composition of bisections  $(\sigma'\sigma)(u) := \sigma'(t\sigma(u))\sigma(u)$ , where the composition on the righthand side is in  $\mathcal{G}$ . With these actions,  $A_{UU} = G_U$  for any translation open set  $\tilde{U}$ .

**Lemma 6.2.** *For each translation open set  $\tilde{V} \subseteq \mathcal{G}_0$  and each  $x \in \mathcal{G}_0$  such that the orbit  $\mathcal{G}_x$  intersects  $\tilde{V}$ , there is a translation open neighbourhood  $\tilde{U}$  of  $x$  such that the evaluation map  $A_{VU} \times \tilde{U} \rightarrow \mathcal{G}(U, V)$  is a diffeomorphism.*

*Proof.* Since  $\tilde{V}$  is a translation neighbourhood,  $s^{-1}(x) \cap t^{-1}(\tilde{V})$  is finite (the intersection of any orbit with a translation neighbourhood is finite and each point has a finite isotropy group.) For each  $g \in s^{-1}(x) \cap t^{-1}(\tilde{V})$ , choose a local bisection  $\tilde{V}_g$  containing  $g$  and such that  $t(\tilde{V}_g) \subseteq \tilde{V}$ . Let  $W_x = \bigcap s(\tilde{V}_g)$  and let  $\tilde{U}$  be a translation neighbourhood of  $x$  such that  $\tilde{U} \subseteq W_x$ . Then the evaluation map clearly defines an embedding  $A_{VU} \times \tilde{U} \rightarrow \mathcal{G}(U, V)$ . This embedding is surjective by the way  $\tilde{U}$  was constructed.  $\square$

Next we consider the relationship between two translation neighbourhoods  $\tilde{U}$  and  $\tilde{V}$  of  $\mathcal{G}_0$ , where one is a subset of the other up to equivalence: that is, such that  $\tilde{U}/\mathcal{G}_1 \subseteq \tilde{V}/\mathcal{G}_1$  in the quotient groupoid  $\mathcal{G}_0/\mathcal{G}_1$ .

**Lemma 6.3.** *Let  $\tilde{U}, \tilde{V} \subseteq \mathcal{G}_0$  be translation open subsets with  $\tilde{U}$  connected and simply connected such that  $\tilde{U}/\mathcal{G}_1 \subseteq \tilde{V}/\mathcal{G}_1$ . Then the evaluation map  $A_{VU} \times \tilde{U} \rightarrow \mathcal{G}(U, V)$  is a diffeomorphism and the canonical left  $G_V$  action on  $A_{VU}$  is free and transitive.*

*Proof.* Since  $\tilde{U}/\mathcal{G}_1 \subseteq \tilde{V}/\mathcal{G}_1$ , the map  $s: s^{-1}(\tilde{U}) \cap t^{-1}(\tilde{V}) \rightarrow \tilde{U}$  is surjective. The previous lemma implies that it is a covering with  $G_V$  as group of deck transformations. Since  $\tilde{U}$  is connected and simply connected, this implies that  $s^{-1}(\tilde{U}) \cap t^{-1}(\tilde{V}) \cong$

$G_V \times \tilde{U}$ , a disjoint union of local bisections which are all diffeomorphic to  $\tilde{U}$ . It follows also immediately that the action of  $G_V$  on  $A_{VU}$  is free and transitive.  $\square$

By interchanging the roles of  $s$  and  $t$  in the previous proof, we obtain the following result.

**Corollary 6.4.** *Let  $\tilde{U}$  be a translation neighbourhood which is connected and simply connected and let  $\tilde{V}$  be a translation neighbourhood such that  $\tilde{U}/\mathcal{G}_1 \subseteq \tilde{V}/\mathcal{G}_1$ . Then  $s^{-1}(\tilde{V}) \cap t^{-1}(\tilde{U})$  consists of a disjoint union of local bisections, each of which is diffeomorphic to  $\tilde{U}$  via  $t$ .*

**Remarks 6.5.** The structure maps of the groupoid allow us to invert surjective local bisections, or invert them on their image when they are not surjective, and compose them when their image and domain of definition match.

**6.2. The Atlas Construction.** We are now ready to construct an orbifold atlas  $\mathfrak{U}$  from each smooth étale groupoid  $\mathcal{G}$  with proper diagonal.

The first thing we need is a collection of orbifold charts as in condition (1) of Definition 4.8. We choose a collection  $\mathcal{U}$  of connected and simply connected translation neighbourhoods that cover the space of objects  $\mathcal{G}_0$ . These translation neighbourhoods will be denoted by  $\tilde{U}_i$ , and their quotients  $\tilde{U}_i/\mathcal{G} = U_i$ . Furthermore, we require that whenever a point  $\bar{x} \in \mathcal{G}_0/\mathcal{G}_1$  lies in the quotients  $U_1$  and  $U_2$  of two translation neighbourhoods ( $\tilde{U}_1$  and  $\tilde{U}_2$  respectively), there is a third translation neighbourhood  $\tilde{V}$  with  $\bar{x} \in V$  and  $V \subseteq U_1 \cap U_2$ ; it is possible to do this because the translation neighbourhoods form a basis for the topology of  $\mathcal{G}_0$  and the spaces are paracompact and Hausdorff.

For each translation neighbourhood  $\tilde{U}$ , we get an *atlas chart*

$$(\tilde{U}, G_U, \rho_U, \pi_U)$$

where  $G_U$  is the set of connected components of  $s^{-1}(\tilde{U}) \cap t^{-1}(\tilde{U})$ ,  $\rho_U: G_U \times \tilde{U} \rightarrow \tilde{U}$  is the action map defined by evaluation and  $\pi_U: \tilde{U} \rightarrow \tilde{U}/\mathcal{G}_1 = U$  is the quotient map.

With these charts, the concrete embeddings are given as follows. For charts  $\tilde{U}_i$  and  $\tilde{U}_j$  with  $U_i \subseteq U_j$  in the quotient space,  $\text{Con}(\mu_{ji})$  is the collection of embeddings obtained by taking composites  $\tilde{U}_i \xrightarrow{s^{-1}} W \xrightarrow{t} \tilde{U}_j$  for each connected component  $W$  of  $s^{-1}(\tilde{U}_i) \cap t^{-1}(\tilde{U}_j)$ . These are effective embeddings since each  $W$  is a local bisection. Note that different local bisections  $W$  may give rise to the same effective embedding in  $\text{Con}(\mu_{ji})$ .

Next, we define the bimodules of embeddings **Abst** as in condition (2) of Definition 4.8. If  $\tilde{U}$  and  $\tilde{V}$  are translation open subsets with  $U \subseteq V$ , define

$$\text{Abst}(\mu_{VU}) = A_{VU},$$

with the actions by  $G_{\tilde{U}}$  and  $G_{\tilde{V}}$  as defined above. These actions are free since  $\mathcal{G}$  is a groupoid, and the action by  $G_{\tilde{V}}$  is transitive as shown in Lemma 6.3.

For translation open sets  $\tilde{U}_i, \tilde{U}_j, \tilde{U}_k$ , with  $U_i \subseteq U_j \subseteq U_k$ , the bimodule transformation  $\alpha_{kji}: A_{\tilde{U}_k \tilde{U}_j} \otimes_{G_{U_j}} A_{\tilde{U}_j \tilde{U}_i} \rightarrow A_{\tilde{U}_k \tilde{U}_i}$  is induced by composition of local bisections:  $(\sigma', \sigma) \mapsto \sigma' \sigma$  as noted in Remark 6.5. Also, for any translation open set  $\tilde{U}$ , the transformation  $\alpha_U: G_{\tilde{U}} \rightarrow A_{\tilde{U} \tilde{U}}$  is the identity, because of the way we

have defined the group. The usual associativity and unit conditions for an internal category imply associativity and unit coherence for the  $\alpha$ s.

We define the oplax transformation  $\rho$  from **Abst** to **Con** as in condition (3) of Definition 4.8 as follows:

$$\rho_{ji} : \rho_j \otimes_{G_j} \text{Abst}(\mu_{ji}) \rightarrow \text{Con}(\mu_{ji}) \otimes_{G_i^{\text{red}}} \rho_i, \quad \rho_j(g_j) \otimes_{G_j} \sigma \rightarrow t \circ (g_j \sigma) \otimes_{G_i^{\text{red}}} e_i^{\text{red}}$$

for all  $g_j \in G_j$  and  $\sigma \in \text{Abst}(\mu_{ji})$ , where  $g_j \sigma$  is constructed as in Section 6.1. A straightforward computation shows that this is well-defined (i.e. it does not depend on the representative chosen for  $\rho_j(g_j) \otimes_{G_j} \sigma$  in  $\rho_j \otimes_{G_j} \text{Abst}(\mu_{ji})$ , that  $\rho_{ji}$  commutes with the actions of  $G_{U_i}$  and  $G_{U_j}$ , and that it is surjective.

Finally, we need to check condition (3b): let us suppose that  $\sigma$  and  $\tau$  are two objects of  $\text{Abst}(\mu_{ji})$ , such that  $\rho_{ji}(e_j^{\text{red}} \otimes \sigma) = \rho_{ji}(e_j^{\text{red}} \otimes \tau)$ . This implies that the following identity holds in  $\text{Con}(\mu_{ji}) \otimes_{G_i^{\text{red}}} \rho_i$ :

$$t \circ \sigma \otimes_{G_i^{\text{red}}} e_i^{\text{red}} = t \circ \tau \otimes_{G_i^{\text{red}}} e_i^{\text{red}},$$

hence we have  $t \circ \sigma = t \circ \tau$ . So we can compose (as bisections)  $\tau$  with the inverse  $\sigma^*$  of  $\sigma$ ; we denote by  $g$  the composition  $\sigma^* \tau \in \mathbf{A}_{\tilde{U}_i, \tilde{U}_i} = G_{\tilde{U}_i}$ . Then we have  $\sigma g = \tau$ , so condition (3b) is satisfied.

**Remark 6.6.** The atlases described in Example 4.15 are constructed via this process from the groupoids described in Example 2.1 by taking the  $U_i$  as translation neighbourhoods.

## 7. EQUIVALENCES

In this section we will define a notion of equivalence for orbifold atlases. This definition generalizes the notion of atlas equivalence for effective orbifolds, and the constructions described in Sections 5 and 6 give us a one-to-one correspondence between equivalence classes of orbifold atlases and Morita equivalence classes of orbifold groupoids.

**7.1. Equivalence of Orbifold Atlases.** Recall that a Satake atlas  $\mathcal{U}$  refines another Satake atlas  $\mathcal{V}$  precisely when the covering by open subsets induced by  $\mathcal{U}$  refines the covering by open subsets induced by  $\mathcal{V}$ , and there are Satake embeddings from the charts in  $\mathcal{U}$  to the charts in  $\mathcal{V}$ . With this definition, two orbifold atlases are equivalent if they have a common refinement. For Satake atlases, the fact that  $\mathcal{U}$  is a refinement of  $\mathcal{V}$  implies that  $\mathcal{U} \cup \mathcal{V}$  can be made into a larger Satake atlas by adding sufficiently many smaller charts to satisfy the local compatibility condition. This addition of smaller charts can be done in a canonical way since any connected component of an invariant subset of a chart inherits a chart structure.

For ineffective orbifolds we need to adjust the definition of refinement so that it will still imply that the union of an atlas with a refinement has a canonical atlas structure. In addition to requiring a refinement of coverings as in the effective case, we will also ask for some compatibility information between charts in the atlas and charts of its refinements. Thus, we will require the existence of certain atlas bimodules connecting charts of the refinement to those of the atlas, and these need to be suitably compatible with the bimodules that make up the two atlas structures. For readers interested in the origin of the compatibility conditions, the

formal framework is as follows (if one just needs the definition, the concrete data and conditions needed are spelled out in Definition 7.1 below):

Let  $\mathfrak{U}$  be an atlas on  $X$  determined by a pair of pseudofunctors and a pseudonatural transformation

$$\mathcal{O}(\mathfrak{U}) \begin{array}{c} \xrightarrow{\text{Abst}_{\mathfrak{U}}} \\ \Downarrow \rho_{\mathfrak{U}} \\ \xrightarrow{\text{Con}_{\mathfrak{U}}} \end{array} \text{GroupMod},$$

and let  $\mathfrak{V}$  be an atlas (on the same space  $X$ ) determined by a pair of pseudofunctors and a pseudonatural transformation

$$\mathcal{O}(\mathfrak{V}) \begin{array}{c} \xrightarrow{\text{Abst}_{\mathfrak{V}}} \\ \Downarrow \rho_{\mathfrak{V}} \\ \xrightarrow{\text{Con}_{\mathfrak{V}}} \end{array} \text{GroupMod}.$$

Then a *refinement module* from  $\mathfrak{U}$  to  $\mathfrak{V}$  is given by the bimodule

$$R: \mathcal{O}(\mathfrak{U}) \rightharpoonup \mathcal{O}(\mathfrak{V})$$

defined by

$$R(V_j, U_i) = \begin{cases} \{*\} & \text{if } U_i \subseteq V_j; \\ \emptyset & \text{else,} \end{cases}$$

together with the structure to make this both a module over  $\text{GroupMod}$  from  $\text{Abst}_{\mathfrak{U}}$  to  $\text{Abst}_{\mathfrak{V}}$ , and also a module over  $\text{GroupMod}$  from  $\text{Con}_{\mathfrak{U}}$  to  $\text{Con}_{\mathfrak{V}}$ . Equivalently, we need the data to construct a pseudofunctor on the bipartite category  $\text{Bipart}(R)$  (also called the *cograph* of  $R$  [St]), which restricts to  $\text{Abst}_{\mathfrak{U}}$  on  $\mathcal{O}(\mathfrak{U})$  and to  $\text{Abst}_{\mathfrak{V}}$  on  $\mathcal{O}(\mathfrak{V})$ , and similarly the data to construct a pseudofunctor on this same bipartite category that restricts to  $\text{Con}_{\mathfrak{U}}$  on  $\mathcal{O}(\mathfrak{U})$  and to  $\text{Con}_{\mathfrak{V}}$  on  $\mathcal{O}(\mathfrak{V})$ .

**Definition 7.1.** Given two orbifold atlases  $\mathfrak{U} = (\mathcal{U}, \text{Abst}_{\mathfrak{U}}, \rho_{\mathfrak{U}})$  and  $\mathfrak{V} = (\mathcal{V}, \text{Abst}_{\mathfrak{V}}, \rho_{\mathfrak{V}})$  for the same topological space  $X$ , we say that  $\mathfrak{U}$  is a *refinement* of  $\mathfrak{V}$  if the following conditions are satisfied:

- (1) The Satake atlas  $\mathcal{U}^{\text{red}}$  is a refinement of the Satake atlas  $\mathcal{V}^{\text{red}}$ .
- (2) For any chart  $\tilde{V}$  in  $\mathcal{V}$  and chart  $\tilde{U}$  in  $\mathcal{U}$  with a point  $x \in U \cap V \subseteq X$ , there is a chart  $\tilde{W}$  in  $\mathcal{U}$  with  $x \in W \subseteq U \cap V$ .
- (3) Whenever  $\tilde{U}_i \in \mathcal{U}$ ,  $\tilde{V}_j \in \mathcal{V}$  and  $U_i \subseteq V_j$  in  $\mathcal{O}(X)$ , there is an atlas bimodule  $A_{ji} = A(V_j, U_i): G_i \rightharpoonup G_j$  with a surjective 2-cell  $\rho_{ji}$  as in the following diagram:

$$\begin{array}{ccc} G_i & \xrightarrow{A(V_j, U_i)} & G_j \\ \rho_i \downarrow & \Downarrow \rho_{ji} & \downarrow \rho_j \\ G_i^{\text{red}} & \xrightarrow{C(V_j, U_i)} & G_j^{\text{red}} \end{array}$$

Here,  $C_{ji} = C(V_j, U_i)$  denotes the module of concrete embeddings  $\lambda: \tilde{U}_i \hookrightarrow \tilde{V}_j$  such that  $\pi_j \circ \lambda = \pi_i$ , where  $\pi_i: \tilde{U}_i \rightarrow U_i$  and  $\pi_j: \tilde{V}_j \rightarrow V_j$  are the quotient maps that are part of the chart structure for  $U_i$  and  $V_j$  respectively.

We require that whenever  $\rho_{ji}(e_j^{\text{red}} \otimes \lambda) = \rho_{ji}(e_j^{\text{red}} \otimes \lambda')$  for  $\lambda, \lambda' \in A_{ji}$ , there is an element  $g \in G_i$  such that  $\lambda \cdot g = \lambda'$ .

- (4) For any  $\mu_{ii'}$  in  $\mathcal{O}(\mathcal{U})$  with  $U_i \subseteq V_j$ , there are bimodule isomorphisms

$$\alpha_{jii'} : A_{ji} \otimes_{G_i} \text{Abst}_{\mathcal{U}}(\mu_{ii'}) \Longrightarrow A_{ji'}$$

and

$$\gamma_{jii'} : C_{ji} \otimes_{G_i^{\text{red}}} \text{Con}_{\mathcal{U}}(\mu_{ii'}) \Longrightarrow C_{ji'}.$$

- (5) For any  $\mu_{j'j}$  in  $\mathcal{O}(\mathcal{V})$  with  $U_i \subseteq V_j$ , there are bimodule isomorphisms

$$\alpha_{j'ji} : \text{Abst}_{\mathcal{V}}(\mu_{j'j}) \otimes_{G_j} A_{ji} \Longrightarrow A_{j'i}$$

and

$$\gamma_{j'ji} : \text{Con}_{\mathcal{V}}(\mu_{j'j}) \otimes_{G_j^{\text{red}}} C_{ji} \Longrightarrow C_{j'i}.$$

- (6) These isomorphisms need to satisfy certain coherence laws and be compatible with the  $\rho_{ii'}$ s,  $\rho_{ji}$ s and  $\rho_{j'i}$ s in the following sense.

First, the  $\alpha_{jii'}$  and  $\gamma_{jii'}$  need to be compatible with the  $\rho_{ii'}$ , the  $\rho_{ji'}$ , and the  $\rho_{ji}$  in the sense that the following composites are equal,

The diagram shows two equivalent commutative structures. On the left, a square with nodes  $G_i$  (top),  $G_{i'}$  (middle left),  $G_j$  (middle right), and  $G_j^{\text{red}}$  (bottom). Arrows:  $G_{i'} \xrightarrow{\text{Abst}_{\mathcal{U}}(\mu_{ii'})} G_i$ ,  $G_i \xrightarrow{A_{ji}} G_j$ ,  $G_{i'} \xrightarrow{A_{ji'}} G_j$ ,  $G_{i'} \xrightarrow{\rho_{i'}} G_j^{\text{red}}$ ,  $G_j \xrightarrow{\rho_j} G_j^{\text{red}}$ ,  $G_j^{\text{red}} \xrightarrow{C_{ji'}} G_j^{\text{red}}$ . A central arrow  $G_{i'} \xrightarrow{\rho_{ji'}} G_j$  is crossed out. A double arrow  $G_i \xrightarrow{\alpha_{jii'}} G_j$  is shown. On the right, the same structure is shown but with an additional node  $G_i^{\text{red}}$  and arrow  $G_i \xrightarrow{\rho_i} G_i^{\text{red}}$ . The central arrow is now  $G_{i'} \xrightarrow{\text{Con}_{\mathcal{U}}(\mu_{ii'})} G_i^{\text{red}}$ . A double arrow  $G_i^{\text{red}} \xrightarrow{\gamma_{jii'}} G_j^{\text{red}}$  is shown. The two diagrams are equated with  $\equiv$ .

Also, the  $\alpha_{j'ji}$  and  $\gamma_{j'ji}$  need to be compatible with the  $\rho_{ji}$ , the  $\rho_{j'i}$  and the  $\rho_{j'j}$  in the sense that the following composites are equal,

The diagram shows two equivalent commutative structures. On the left, a square with nodes  $G_i$  (bottom left),  $G_j$  (top),  $G_{j'}$  (bottom right), and  $G_{j'}^{\text{red}}$  (middle right). Arrows:  $G_i \xrightarrow{A_{ji}} G_j$ ,  $G_j \xrightarrow{\text{Abst}_{\mathcal{V}}(\mu_{j'j})} G_{j'}$ ,  $G_i \xrightarrow{A_{j'i}} G_{j'}$ ,  $G_i \xrightarrow{\rho_i} G_i^{\text{red}}$ ,  $G_j \xrightarrow{\rho_j} G_j^{\text{red}}$ ,  $G_{j'} \xrightarrow{\rho_{j'}} G_{j'}^{\text{red}}$ ,  $G_{j'}^{\text{red}} \xrightarrow{C_{j'i}} G_{j'}^{\text{red}}$ . A central arrow  $G_i \xrightarrow{\rho_{ji}} G_{j'}$  is crossed out. A double arrow  $G_i \xrightarrow{\alpha_{j'ji}} G_{j'}$  is shown. On the right, the same structure is shown but with an additional node  $G_j^{\text{red}}$  and arrow  $G_j \xrightarrow{\rho_j} G_j^{\text{red}}$ . The central arrow is now  $G_i \xrightarrow{C_{ji}} G_j^{\text{red}}$ . A double arrow  $G_j^{\text{red}} \xrightarrow{\gamma_{j'ji}} G_{j'}^{\text{red}}$  is shown. The two diagrams are equated with  $\equiv$ .

For arrows  $\mu_{ii'}$ ,  $\mu_{ii''} \in \mathcal{O}(\mathcal{U})$  and  $\mu_{j'j}$ ,  $\mu_{j''j'} \in \mathcal{O}(\mathcal{V})$  with  $U_i \subseteq V_j$ , the following generalized associativity pentagons need to commute:

$$\begin{array}{ccc}
& A_{ji} \otimes (\text{Abst}_{\mathfrak{U}}(\mu_{ii'}) \otimes \text{Abst}_{\mathfrak{U}}(\mu_{i'i''})) & \\
& \sim \swarrow \quad \searrow & \\
(A_{ji} \otimes \text{Abst}_{\mathfrak{U}}(\mu_{ii'})) \otimes \text{Abst}_{\mathfrak{U}}(\mu_{i'i''}) & & A_{ji} \otimes \text{Abst}_{\mathfrak{U}}(\mu_{ii''}) \\
\downarrow \alpha_{jii'} \otimes \text{Abst}_{\mathfrak{U}}(\mu_{i'i''}) & & \downarrow \alpha_{jii''} \\
A_{ji'} \otimes \text{Abst}_{\mathfrak{U}}(\mu_{i'i''}) & \xrightarrow{\alpha_{j'i'i''}} & A_{j'i''}
\end{array}$$

$$\begin{array}{ccc}
& \text{Abst}_{\mathfrak{V}}(\mu_{j'j}) \otimes (A_{ji} \otimes \text{Abst}_{\mathfrak{U}}(\mu_{ii'})) & \\
& \sim \swarrow \quad \searrow & \\
(\text{Abst}_{\mathfrak{V}}(\mu_{j'j}) \otimes A_{ji}) \otimes \text{Abst}_{\mathfrak{U}}(\mu_{ii'}) & & \text{Abst}_{\mathfrak{V}}(\mu_{j'j}) \otimes A_{ji'} \\
\downarrow \alpha_{j'ji} \otimes \text{Abst}_{\mathfrak{U}}(\mu_{ii'}) & & \downarrow \alpha_{j'ji'} \\
A_{j'i} \otimes \text{Abst}_{\mathfrak{U}}(\mu_{ii'}) & \xrightarrow{\alpha_{j'i'i'}} & A_{j'i'}
\end{array}$$

$$\begin{array}{ccc}
& \text{Abst}_{\mathfrak{V}}(\mu_{j''j'}) \otimes (\text{Abst}_{\mathfrak{V}}(\mu_{j'j}) \otimes A_{ji}) & \\
& \sim \swarrow \quad \searrow & \\
(\text{Abst}_{\mathfrak{V}}(\mu_{j''j'}) \otimes \text{Abst}_{\mathfrak{V}}(\mu_{j'j})) \otimes A_{ji} & & \text{Abst}_{\mathfrak{V}}(\mu_{j''j'}) \otimes A_{j'i} \\
\downarrow \alpha_{j''j'j} \otimes A_{ji} & & \downarrow \alpha_{j''j'i} \\
\text{Abst}_{\mathfrak{V}}(\mu_{j''j}) \otimes A_{ji} & \xrightarrow{\alpha_{j''ji}} & A_{j''i}
\end{array}$$

Whenever  $U_i \subseteq V_j$ , the following unit coherence diagrams need to commute:

$$\begin{array}{ccc}
A_{ji} \otimes G_i & \xrightarrow{A_{ji} \otimes \alpha_i} & A_{ji} \otimes \text{Abst}_{\mathfrak{U}}(\mu_{ii}) \\
& \searrow \sim & \downarrow \alpha_{jii} \\
& & A_{ji}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
G_j \otimes A_{ji} & \xrightarrow{\alpha_j \otimes A_{ji}} & \text{Abst}_{\mathfrak{V}}(\mu_{jj}) \otimes A_{ji} \\
& \searrow \sim & \downarrow \alpha_{jji} \\
& & A_{ji}
\end{array}$$

where the  $\sim$  indicates the canonical isomorphism.

Analogous associativity and unit coherence conditions apply to the  $\gamma_{jii'}$  and  $\gamma_{j'ji}$ .

With this definition, it is easy to check that refinements still have the following transitive property.

**Lemma 7.2.** *Suppose we have orbifold atlases  $\mathfrak{U}$ ,  $\mathfrak{V}$ , and  $\mathfrak{W}$  for the same topological space  $X$ , and  $\mathfrak{U}$  is a refinement of  $\mathfrak{V}$  and  $\mathfrak{V}$  is a refinement of  $\mathfrak{W}$ . Then  $\mathfrak{U}$  is a refinement of  $\mathfrak{W}$ .*

The notion of atlas equivalence now generalizes as follows:

**Definition 7.3.** Two orbifold atlases  $\mathfrak{V}$  and  $\mathfrak{W}$  for the same underlying space  $X$  are *equivalent* if they have a common refinement  $\mathfrak{U}$ .

This relation is clearly symmetric and reflexive. It is also transitive; this will follow from Theorems 7.8 and 7.13 proved below (and the fact that being Morita equivalent is an equivalence relation).

**Remarks 7.4.** (1) The argument given in Lemma 4.12 applies here showing that each  $\rho_{ji}$  in part (3) of Definition 7.1 gives rise to a module map  $\tilde{\rho}_{ji}: \mathbf{A}_{ji} \rightarrow \mathbf{C}_{ji}$ .

- (2) It is clear from the set-up of the definition of refinement that the union of an orbifold atlas  $\mathfrak{V}$  with a refinement  $\mathfrak{U}$  gives rise to a canonical new atlas structure, except for the fact that any charts which occur in both atlases will occur twice in the new structure. Because of this, we obtain the following generalization of the strong compatibility result for atlas charts (Lemma 4.19):

*For any charts  $\tilde{U}_1$  in  $\mathcal{U}$  and  $\tilde{V}_2, \tilde{V}_3$  in  $\mathcal{V}$  with  $U_1 \subseteq V_3$  and  $V_2 \subseteq V_3$  and for any points  $x_1 \in \tilde{U}_1$ ,  $y_2 \in \tilde{V}_2$  and  $y_3 \in \tilde{V}_3$  and abstract embeddings  $\lambda_{31} \in \mathbf{A}_{31}$  and  $\lambda_{32} \in \mathbf{Abst}_{\mathfrak{V}}(\mu_{32})$ , such that  $\tilde{\rho}_{31}(\lambda_{31})(x_1) = y_3$  and  $\tilde{\rho}_{32}(\lambda_{32})(y_2) = y_3$ , there is a chart  $\tilde{U}_0$  in  $\mathcal{U}$  with a point  $x_0 \in \tilde{U}_0$  and abstract embeddings  $\kappa_{10} \in \mathbf{Abst}_{\mathfrak{U}}(\mu_{10})$  and  $\kappa_{20} \in \mathbf{A}_{20}$ , such that  $\tilde{\rho}_{10}(\kappa_{10})(x_0) = x_1$ ,  $\tilde{\rho}_{20}(\kappa_{20})(x_0) = y_2$  and  $\alpha_{310}(\lambda_{31} \otimes \kappa_{10}) = \alpha_{320}(\lambda_{32} \otimes \kappa_{20})$ .*

**7.2. Equivalence of Smooth Groupoids.** In this section we summarize the description of equivalence of proper étale groupoids in terms of Hilsum-Skandalis maps [H, Pr].

**Definition 7.5.** Let  $\mathcal{G}$  be a topological groupoid. A *right  $\mathcal{G}$ -bundle* over a manifold  $X$  is a manifold  $M$  with smooth maps

$$\begin{array}{ccc} M & \xrightarrow{\varepsilon} & X \\ \tau \downarrow & & \\ & \mathcal{G}_0 & \end{array}$$

and a right  $\mathcal{G}$ -action  $\mu$  on  $M$ , with anchor map  $\tau: M \rightarrow \mathcal{G}_0$ , such that  $\varepsilon(mg) = \varepsilon(m)$  ( $\mathcal{G}$  acts on the fibres of  $\varepsilon$ ) and  $\tau(mg) = s(g)$  for any  $m \in M$  and any  $g \in \mathcal{G}_1$  with  $\tau(m) = t(g)$ .

Such a bundle  $M$  is *principal* if

- (1)  $\varepsilon$  is a surjective submersion, and
- (2) the map  $(pr_1, \mu): M \times_{\mathcal{G}_0} \mathcal{G}_1 \rightarrow M \times_X M$ , sending  $(m, g)$  to  $(m, mg)$ , is a diffeomorphism.

**Definition 7.6.** A *Hilsum-Skandalis map*  $\mathcal{G} \rightarrow \mathcal{H}$  is represented by a right  $\mathcal{G}$ -bundle  $M$  over  $\mathcal{H}_0$ ,  $\mathcal{G}_0 \xleftarrow{\tau} M \xrightarrow{\varepsilon} \mathcal{H}_0$ , which is also a principal left  $\mathcal{H}$ -bundle over  $\tau$ , and such that the left and right actions commute. So we have that

$$\begin{aligned} \tau(hm) &= \tau(m) \text{ and } \varepsilon(mg) = \varepsilon(m) \\ h(mg) &= (hm)g \\ \tau(mg) &= s(g) \text{ and } \varepsilon(hm) = t(h) \end{aligned}$$

for any  $m \in M$ ,  $g \in \mathcal{G}_1$  and  $h \in \mathcal{H}_1$  with  $s(h) = \varepsilon(m)$  and  $t(g) = \tau(m)$ .

Moreover, since the  $\mathcal{H}$ -bundle is principal,  $\varepsilon$  is a surjective submersion, and the map  $\mathcal{H}_1 \times_{\mathcal{H}_0} M \rightarrow M \times_{\mathcal{G}_0} M$  is a diffeomorphism.

**Definition 7.7.** A Hilsum-Skandalis map is a *Morita equivalence* when it is both a principal left  $\mathcal{G}$ -bundle and a principal right  $\mathcal{H}$ -bundle.

Two proper étale groupoids are Morita equivalent if there is a Morita equivalence between them. Note that in particular, if  $\mathcal{G}$  and  $\mathcal{H}$  are Morita equivalent, we get an induced homeomorphism between their quotient spaces  $\mathcal{G}_0/\mathcal{G}_1 \cong \mathcal{H}_0/\mathcal{H}_1$ .

**7.3. Morita Equivalence Implies Atlas Equivalence.** The goal of this section is to prove the following:

**Theorem 7.8.** *Let  $X$  be a space, and  $\mathcal{G}$  and  $\mathcal{H}$  be proper étale groupoids such that both  $\mathcal{G}_0/\mathcal{G}_1$  and  $\mathcal{H}_0/\mathcal{H}_1$  are homeomorphic to  $X$ . If  $\mathcal{G}$  and  $\mathcal{H}$  are Morita equivalent, and  $\mathfrak{V}$  and  $\mathfrak{W}$  are orbifold atlases for  $X$  constructed from the translation neighbourhoods in  $\mathcal{G}$  and  $\mathcal{H}$  respectively, then  $\mathfrak{V}$  and  $\mathfrak{W}$  are equivalent in the sense of Definition 7.3.*

This result implies that any two atlases obtained from the same groupoid (as in Section 6.2) are equivalent as orbifold atlases:

**Corollary 7.9.** *If  $\mathcal{G}$  is a proper étale groupoid and  $\mathfrak{V}$  and  $\mathfrak{W}$  are two orbifold atlases constructed from translation neighbourhoods in  $\mathcal{G}$ , then  $\mathfrak{V}$  and  $\mathfrak{W}$  are equivalent atlases for the quotient space  $\mathcal{G}_0/\mathcal{G}_1$ .*

We will prove Theorem 7.8 in several stages. We begin by fixing notation. Throughout this subsection,  $\mathcal{G}$  and  $\mathcal{H}$  will denote proper étale groupoids with a Hilsum-Skandalis Morita equivalence

$$M: \mathcal{G} \rightrightarrows \mathcal{H}$$

with bundle maps  $\mathcal{G}_0 \xleftarrow{\tau} M \xrightarrow{\varepsilon} \mathcal{H}_0$ . Let  $X$  be a space that is homeomorphic to  $\mathcal{G}_0/\mathcal{G}_1 \cong \mathcal{H}_0/\mathcal{H}_1$ . Furthermore,  $\mathfrak{V}$  and  $\mathfrak{W}$  will denote induced atlases consisting of translation neighbourhoods for  $X$  in  $\mathcal{G}$  and  $\mathcal{H}$  respectively. Note that the maps from the charts in  $\mathfrak{V}$  into the underlying space  $X$  are obtained as restrictions of the composition of the projection map  $\mathcal{G}_0 \rightarrow \mathcal{G}_0/\mathcal{G}_1$  with the isomorphism  $\mathcal{G}_0/\mathcal{G}_1 \cong X$ ; similarly for  $\mathfrak{W}$ .

Since  $\mathfrak{V}$  consists of translation neighbourhoods in  $\mathcal{G}$ , we can write  $s^{-1}(\tilde{V}) \cap t^{-1}(\tilde{V}) \cong G_V \times \tilde{V}$  for all  $\tilde{V} \in \mathcal{V}$ , and similarly for  $\mathfrak{W}$ :  $s^{-1}(\tilde{W}) \cap t^{-1}(\tilde{W}) \cong H_W \times \tilde{W}$  for all  $\tilde{W} \in \mathcal{W}$ . Note that the structure groups  $G_V$  and  $H_W$  are all finite and discrete. We will prove below that for any  $m \in M$  we can choose an open neighbourhood  $S_m$  containing  $m$ , such that  $\tau|_{S_m}: S_m \rightarrow \tau(S_m)$  and  $\varepsilon|_{S_m}: S_m \rightarrow \varepsilon(S_m)$  are diffeomorphisms and  $\tau(S_m)$  and  $\varepsilon(S_m)$  are invariant subsets of  $\tilde{V}$  and  $\tilde{W}$  respectively. Explicitly, this means that for each  $g \in G_V$  either  $g(\tau(S_m)) = \tau(S_m)$  or  $g(\tau(S_m)) \cap \tau(S_m) = \emptyset$ , and analogously for each  $h \in H_W$  and  $\varepsilon(S_m)$ . We will call any such neighbourhood  $S_m$  an *invariant local bisection* of  $M$ .

Our plan is to show that these invariant local bisections can be used to create an atlas  $\mathfrak{U}$  which is a common refinement of  $\mathfrak{V}$  and  $\mathfrak{W}$ . Our first proposition shows that there are enough of these invariant local bisections.

**Proposition 7.10.** *The invariant local bisections form a basis for the topology on the space  $\tau^{-1}(\bigcup_{\tilde{V} \in \mathcal{V}} \tilde{V}) \cap \varepsilon^{-1}(\bigcup_{\tilde{W} \in \mathcal{W}} \tilde{W}) \subseteq M$ .*

*Proof.* First, since  $\mathcal{G}$  is étale and  $M \xrightarrow{\varepsilon} \mathcal{H}_0$  is a right principal  $\mathcal{G}$ -bundle, the map  $\varepsilon$  is a submersion with discrete fibers, so  $\varepsilon$  is étale. Similarly,  $\tau: M \rightarrow \mathcal{G}_0$  is étale (it



is a left principal  $\mathcal{H}$ -bundle and  $\mathcal{H}$  is étale). Hence, open subsets  $S \subseteq M$  for which both the restriction of  $\varepsilon$  and the restriction of  $\tau$  are diffeomorphisms form a basis for the topology.

Since  $\varepsilon$  is a submersion, we may consider the pullback groupoid

$$\varepsilon^*\mathcal{H} := (M \times_{\varepsilon} \mathcal{H}_1 \times_{\varepsilon} M \rightrightarrows M)$$

and this is again an étale groupoid since  $s$ ,  $t$  and  $\varepsilon$  are étale maps. Also, the functor

$$\begin{array}{ccc} M \times_{\varepsilon} \mathcal{H}_1 \times_{\varepsilon} M & \xrightarrow{\pi} & \mathcal{H}_1 \\ \Downarrow & & \Downarrow \\ M & \xrightarrow{\varepsilon} & \mathcal{H}_0 \end{array}$$

is clearly a Morita equivalence. Thus, for any open  $S \subseteq M$  such that  $\varepsilon|_S$  is a diffeomorphism, with  $W = \varepsilon(S)$ , the induced map between the restricted groupoids,

$$\begin{array}{ccc} S \times_{\varepsilon} \mathcal{H}_1 \times_{\varepsilon} S & \xrightarrow{\pi} & \mathcal{H}_1|_W \\ \Downarrow & & \Downarrow \\ S & \xrightarrow{\varepsilon} & W \end{array}$$

is an isomorphism of smooth groupoids. Hence,  $S$  is a translation open subset for  $\varepsilon^*\mathcal{H}$  if and only if  $W$  is a translation open subset for  $\mathcal{H}$ . So the translation open subsets form a basis for the topology of  $M$ .

By symmetry, similar results hold for  $\tau$  and  $\mathcal{G}$ . Furthermore, by bi-principality of the Hilsum-Skandalis bundle, there is a canonical isomorphism of Lie groupoids,  $\varepsilon^*\mathcal{H} \cong \tau^*\mathcal{G}$ , given by  $(m', h, m) \mapsto (m', g, m)$ , where  $g$  and  $h$  uniquely determine each other through the equation  $m'g = hm$ . So any open subset  $S \subseteq M$  is translation open for  $\varepsilon^*\mathcal{H}$  if and only if it is translation open for  $\tau^*\mathcal{G}$ . This gives us the required result.  $\square$

Now fix any collection  $\{S_i\}$  of invariant local bisections which forms a basis for the topology on  $\tau^{-1}(\bigcup_{\tilde{V} \in \mathcal{V}} \tilde{V}) \cap \varepsilon^{-1}(\bigcup_{\tilde{W} \in \mathcal{W}} \tilde{W}) \subseteq M$ . Then we define the atlas  $\mathfrak{U}$  by taking all the charts  $U_i := \tau(S_i)$ . Note that each  $U_i$  is a translation neighbourhood in  $\mathcal{G}$ , so we obtain the remainder of the atlas structure from  $\mathcal{G}$  as described in Section 6.2. In particular, if  $S_i$  is constructed around a point  $m \in \tau^{-1}(V) \cap \varepsilon^{-1}(W)$  (as in Proposition 7.10), then  $U_i$  gets the structure group  $G_{U_i} \leq G_V$  consisting of all the elements  $g \in G_V$  such that  $g(U_i) = U_i$ .

**Remarks 7.11.** We have arbitrarily chosen to define our atlas structure in terms of  $\mathcal{G}$ ; we would obtain the same structure groups if we defined it in terms of  $\mathcal{H}$ , since the subgroup  $G_U$  has a free and transitive right action on the connected components of this subspace, whereas a corresponding subgroup  $H_U$  has a free and transitive left action on the set of connected components. These actions commute in the sense that for any connected component  $C$ ,  $h(Cg) = (hC)g$ , and so the groups are isomorphic.

The following will complete the proof of Theorem 7.8.

**Proposition 7.12.** *The atlas  $\mathfrak{U}$  is a common refinement for  $\mathfrak{V}$  and  $\mathfrak{W}$ .*

*Proof.* The atlas bimodules  $A_{ji}^{\mathfrak{V}}: G_{U_i} \dashrightarrow G_{V_j}$  and  $C_{ji}^{\mathfrak{V}}: G_{U_i}^{\text{red}} \dashrightarrow G_{V_j}^{\text{red}}$ , together with the required module homomorphisms  $\rho_{ji}$ ,  $\alpha_{jii'}$  and  $\gamma_{jii'}$  can be obtained from the structure of the groupoid  $\mathcal{G}$  in the same way as the atlas structure of  $\mathfrak{U}$  is defined, following the method of Section 6.2. The required family of coherent isomorphisms is provided by  $\mathcal{G}$  as well: in fact, all the structure is there to make  $\mathfrak{U} \cup \mathfrak{V}$  an orbifold atlas, since we chose to use  $\mathcal{G}$  in defining  $\mathfrak{U}$ .

To make  $\mathfrak{U}$  a refinement for  $\mathfrak{W}$ , we need to define atlas bimodules  $A_{ki}^{\mathfrak{W}}: G_{U_i} \dashrightarrow H_{W_k}$  and  $C_{ki}^{\mathfrak{W}}: G_{U_i}^{\text{red}} \dashrightarrow H_{W_k}^{\text{red}}$  together with  $\rho_{ki}$  whenever the map  $\tau: \varepsilon^{-1}(\widetilde{W}_k) \cap \tau^{-1}(U_i) \rightarrow U_i$  is surjective (that is, whenever  $U_i$  is a subset of the chart  $\tau(\varepsilon^{-1}(\widetilde{W}_k))$  in  $\mathcal{G}$ , corresponding to the chart  $\widetilde{W}_k$  of  $\mathcal{H}$ ). We define  $A_{ki}^{\mathfrak{W}}$  as the set of connected components of  $\varepsilon^{-1}(\widetilde{W}_k) \cap \tau^{-1}(U_i) \subseteq M$ ; the group  $H_{W_k}$  acts freely and transitively and  $G_{U_i}$  acts freely on this set, so we obtain an atlas bimodule. The bimodule  $C_{ki}^{\mathfrak{W}}$  on the concrete level is obtained as follows: for each component  $T \subseteq \varepsilon^{-1}(\widetilde{W}_k) \cap \tau^{-1}(U_i)$ , we define the concrete embedding  $\lambda_T$  as the composite

$$U_i \xrightarrow{(\tau|_T)^{-1}} T \xrightarrow{\varepsilon} \widetilde{W}_k.$$

and we define  $C_{ki}^{\mathfrak{W}}$  as the set of all such  $\lambda_T$ 's. The 2-cell  $\rho_{ki}$  is defined by sending  $e \otimes T$  to  $\lambda_T \otimes e$ .

For  $\mu_{k'k}$  in  $\mathcal{O}(\mathcal{W})$ , we define the isomorphism  $\alpha_{k'ki}: \text{Abst}_{\mathfrak{W}}(\mu_{k'k}) \otimes_{H_{W_k}} A_{ki}^{\mathfrak{W}} \rightarrow A_{k'i}^{\mathfrak{W}}$  as follows. Recall that the elements of  $\text{Abst}_{\mathfrak{W}}(\mu_{k'k})$  are the connected components of  $s^{-1}(\widetilde{W}_k) \cap t^{-1}(\widetilde{W}_{k'})$ . Note that the map  $\varepsilon$  restricted to any component  $T$  of  $\varepsilon^{-1}(\widetilde{W}_k) \cap \tau^{-1}(U_i)$  is a diffeomorphism, as is the source map when restricted to any component of  $s^{-1}(\widetilde{W}_k) \cap t^{-1}(\widetilde{W}_{k'})$ . So the right action of  $\mathcal{H}$  on  $M$  induces a well-defined map  $\text{Abst}_{\mathfrak{W}}(\mu_{k'k}) \times A_{ki}^{\mathfrak{W}} \rightarrow A_{k'i}^{\mathfrak{W}}$ . Furthermore, this is well-defined with respect to the action of  $H_{W_k}$  so that this factors through the required isomorphism  $\alpha_{k'ki}: \text{Abst}_{\mathfrak{W}}(\mu_{k'k}) \otimes_{H_{W_k}} A_{ki}^{\mathfrak{W}} \rightarrow A_{k'i}^{\mathfrak{W}}$ . The  $\gamma_{k'ki}$  are defined by composition as usual, and the required diagrams involving the  $\rho_{ki}$  are easily seen to commute.

A straightforward calculation shows that these  $\alpha_{k'ki}$  also satisfy the other required coherence conditions. This can also be seen by the following observation: the atlas  $\mathfrak{U}$  can be viewed as an atlas induced by  $\mathcal{H}$  (rather than by  $\mathcal{G}$  as constructed above), by viewing each  $U_i$  as embedded in  $\widetilde{W}_k$  by the embedding  $\lambda_T$ , and considering the structure group to be  $H_{U_i}$  via an isomorphism of this group with  $G_{U_i}$  as in Remark 7.11. The abstract and concrete modules defined by  $\mathcal{G}$  can be translated into abstract and concrete modules defined by  $\mathcal{H}$  through the actions of both groupoids on the invariant local bisections of  $M$ .  $\square$

**7.4. Atlas Equivalence Implies Morita Equivalence.** In this section, we prove the converse of Theorem 7.8.

**Theorem 7.13.** *If  $\mathfrak{V}$  and  $\mathfrak{W}$  are equivalent atlases for a space  $X$ , then the induced groupoids  $\mathcal{G}(\mathfrak{V})$  and  $\mathcal{G}(\mathfrak{W})$  are Morita equivalent.*

*Proof.* Since  $\mathfrak{V}$  and  $\mathfrak{W}$  are equivalent, they have a common refinement  $\mathfrak{U}$  as in Definition 7.1. Let  $I, J$  and  $K$  be the index sets for  $\mathcal{U}, \mathcal{V}$  and  $\mathcal{W}$  respectively. Write  $A_{ji}^{\mathfrak{U}}$  and  $C_{ji}^{\mathfrak{U}}$  (respectively,  $A_{ji}^{\mathfrak{V}}$  and  $C_{ji}^{\mathfrak{V}}$ ) for the atlas bimodules establishing  $\mathfrak{U}$  as a refinement of  $\mathfrak{V}$  (respectively, of  $\mathfrak{W}$ ). We will use these bimodules to construct a

Morita equivalence of groupoids. This will be given by a Hilsum-Skandalis map

$$M(\mathfrak{U}): \mathcal{G}(\mathfrak{V}) \rightharpoonup \mathcal{G}(\mathfrak{W}),$$

with surjective submersions  $\mathcal{G}(\mathfrak{V})_0 \xleftarrow{\tau} M(\mathfrak{U}) \xrightarrow{\varepsilon} \mathcal{G}(\mathfrak{W})_0$ , such that the actions induce diffeomorphisms

$$M(\mathfrak{U}) \times_{\mathcal{G}(\mathfrak{W})_0} \mathcal{G}(\mathfrak{W})_1 \cong M(\mathfrak{U}) \times_{\mathcal{G}(\mathfrak{V})_0} M(\mathfrak{U})$$

and

$$M(\mathfrak{U}) \times_{\mathcal{G}(\mathfrak{V})_0} \mathcal{G}(\mathfrak{V})_1 \cong M(\mathfrak{U}) \times_{\mathcal{G}(\mathfrak{W})_0} M(\mathfrak{U}).$$

Note that since the charts  $\mathcal{V}$  of  $\mathfrak{V}$  have index set  $J$ ,  $\mathcal{G}(\mathfrak{V})_0 = \coprod_{j \in J} \widetilde{V}_j$ ; and similarly,  $\mathcal{G}(\mathfrak{W})_0 = \coprod_{k \in K} \widetilde{W}_k$ .

We will construct the space  $M(\mathfrak{U})$  by constructing subspaces of the form

$$M(\mathfrak{U})_{jk} = \tau^{-1}(\widetilde{V}_j) \cap \varepsilon^{-1}(\widetilde{W}_k),$$

together with the maps

$$\widetilde{V}_j \xleftarrow{\tau} M(\mathfrak{U})_{jk} \xrightarrow{\varepsilon} \widetilde{W}_k.$$

The space  $M(\mathfrak{U})_{jk}$  is constructed as a quotient of the space  $\coprod_{i \in I} \mathbf{A}_{ji}^{\mathfrak{V}} \times \widetilde{U}_i \times \mathbf{A}_{ki}^{\mathfrak{W}}$ , where we give the modules  $\mathbf{A}_{ji}^{\mathfrak{V}}$  and  $\mathbf{A}_{ki}^{\mathfrak{W}}$  the discrete topology (they are empty whenever  $U_i \not\subseteq V_j$ , respectively  $U_i \not\subseteq W_k$ ). The equivalence relation  $\sim$  on  $\coprod \mathbf{A}_{ji}^{\mathfrak{V}} \times \widetilde{U}_i \times \mathbf{A}_{ki}^{\mathfrak{W}}$  is generated by

$$(\lambda_{ji}, \widetilde{\rho}_{ii'}(\nu_{ii'})(x), \lambda_{ki}) \sim (\alpha_{jii'}(\lambda_{ji} \otimes \nu_{ii'}), x, \alpha_{kii'}(\lambda_{ki} \otimes \nu_{ii'}))$$

for any  $\nu_{ii'} \in \mathbf{Abst}_{\mathfrak{U}}(\mu_{ii'})$ . (Note that this equivalence relation is of the same form as the relation used to define the arrow spaces of the atlas groupoids.) Then,

$$M(\mathfrak{U})_{jk} := \left( \coprod_{\substack{\widetilde{U}_i \in \mathcal{U} \\ U_i \subseteq \widetilde{V}_j \cap \widetilde{W}_k}} \mathbf{A}_{ji}^{\mathfrak{V}} \times \widetilde{U}_i \times \mathbf{A}_{ki}^{\mathfrak{W}} \right) / \sim.$$

The map  $\tau: M(\mathfrak{U})_{jk} \rightarrow \widetilde{V}_j$  is defined by  $[\lambda_{ji}, x, \lambda_{ki}] \mapsto \widetilde{\rho}_{ji}(\lambda_{ji})(x)$  and the map  $\varepsilon: M(\mathfrak{U})_{jk} \rightarrow \widetilde{W}_k$  is defined by  $[\lambda_{ji}, x, \lambda_{ki}] \mapsto \widetilde{\rho}_{ki}(\lambda_{ki})(x)$  (with  $\widetilde{\rho}_{ji}$  and  $\widetilde{\rho}_{ki}$  as described in Remark 7.4(1)). Since  $\mathfrak{U}$  is a refinement, these maps are surjective local diffeomorphisms and in particular surjective submersions as required.

We define the manifold  $M(\mathfrak{U}) = \coprod_{j,k} M(\mathfrak{U})_{jk}$ .

The right action of  $\mathcal{G}(\mathfrak{V})$  and the left action of  $\mathcal{G}(\mathfrak{W})$  are defined in a way analogous to composition in atlas groupoids. Let  $g \in \mathcal{G}(\mathfrak{V})_1$  with  $s(g) \in \widetilde{V}_{j'}$  and  $t(g) \in \widetilde{V}_j$ , and let  $(\lambda_{ji}, x, \lambda_{ki})$  represent an element of  $M(\mathfrak{U})_{jk}$  with  $\tau([\lambda_{ji}, x, \lambda_{ki}]) = \widetilde{\rho}_{ji}(\lambda_{ji})(x) = t(g)$ . Then  $g \in \mathcal{G}(\mathfrak{V})_1$  is represented by a triple  $(\theta_{j'j''}, y, \theta_{jj''})$  with  $y \in \widetilde{V}_{j''}$ ,  $\theta_{j'j''} \in \mathbf{Abst}_{\mathfrak{V}}(\mu_{j'j''})$  and  $\theta_{jj''} \in \mathbf{Abst}_{\mathfrak{V}}(\mu_{jj''})$ . Now  $t(g) = \widetilde{\rho}_{jj''}(\theta_{jj''})(y)$ , so  $\widetilde{\rho}_{jj''}(\theta_{jj''})(y) = \widetilde{\rho}_{ji}(\lambda_{ji})(x)$ .

By Remark 7.4(2), this implies that there are a chart  $\widetilde{U}_{i'}$  in  $\mathcal{U}$ , a point  $z \in \widetilde{U}_{i'}$ , an abstract embedding  $\nu_{ii'} \in \mathbf{Abst}_{\mathfrak{U}}(\mu_{ii'})$  and an abstract embedding  $\lambda_{j''i'} \in \mathbf{A}_{j''i'}^{\mathfrak{V}}$  such that  $\widetilde{\rho}_{ii'}(\nu_{ii'})(z) = x$ ,  $\widetilde{\rho}_{j''i'}(\lambda_{j''i'})(z) = y$  and  $\alpha_{jj''i'}(\theta_{jj''} \otimes \lambda_{j''i'}) = \alpha_{jii'}(\lambda_{ji} \otimes \nu_{ii'})$ . Hence,  $(\lambda_{ji}, x, \lambda_{ki}) \sim (\alpha_{jii'}(\lambda_{ji} \otimes \nu_{ii'}), z, \alpha_{kii'}(\lambda_{ki} \otimes \nu_{ii'}))$ .

Then the right action of  $g$  on the class of the point  $(\lambda_{ji}, x, \lambda_{ki})$  is represented by  $(\lambda_{ki}, x, \lambda_{ji}) \cdot g := (\alpha_{j'j''i'}(\theta_{j'j''} \otimes \lambda_{j''i'}), z, \alpha_{kii'}(\lambda_{ki} \otimes \nu_{ii'}))$ . It can be verified that this is independent of the choice of representatives  $(\lambda_{ji}, x, \lambda_{ki})$  in  $M(\mathfrak{U})_{jk}$  and  $(\theta_{j'j''}, y, \theta_{jj''})$  in  $\mathcal{G}(\mathfrak{V})_1$ .

The left action by  $\mathcal{G}(\mathfrak{W})$  on  $M(\mathfrak{U})$  is defined in a similar (but dual) fashion.

It is a straightforward calculation to check that this satisfies all the conditions to be a Morita equivalence.  $\square$

So we conclude that the notion of orbifold defined in terms of orbifold atlases and atlas equivalences as presented in this paper corresponds to the notion of orbifold defined in terms of proper étale groupoids and Morita equivalence.

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